Quantum Lévy process and a possible definition of quantum Brownian motion

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Fock space and probability

Symmetric and anti-symmetric tensor products of Hilbert space

$H \longrightarrow$ Hilbert space

$H^{\otimes n} \longrightarrow$ n-fold tensor product of H

$S_n \longrightarrow$ Group of permutation of n objects

Let $u_1, u_2, \cdots u_n \in H$

n-fold symmetric/Bosonic tensor product: $H^{\vee n}$

Define

$E(u_1 \otimes u_2 \otimes \cdots u_n) := \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \cdots \otimes u_{\sigma^{-1}(n)}$

$$H^{\vee n} := E(H^{\otimes n})$$

n-fold anti-symmetric/Fermionic tensor product: $H^{\wedge n}$

Define

$F(u_1 \otimes u_2 \otimes \cdots u_n) := \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma)\ u_{\sigma^{-1}(1)} \otimes u_{\sigma^{-1}(2)} \cdots \otimes u_{\sigma^{-1}(n)}$
where $\epsilon(\sigma)$ is the signature of σ

$$H^{\wedge n} := F(H^{\otimes n})$$

Symmetric and anti-symmetric tensor products of Hilbert space

$H \longrightarrow L^2(\mathbb{R})$

$S_n \longrightarrow$ Group of permutation of n objects

$H^{\otimes n} = L^2(\mathbb{R}^n)$

n-fold symmetric/Bosonic tensor product: $H^{\vee n}$

$H^{\vee n} := \{f \in L^2(\mathbb{R}^n) : \quad f$ is symmetric in n variables$\}$

n-fold anti-symmetric/Fermionic tensor product: $H^{\wedge n}$

$H^{\wedge n} := \{f \in L^2(\mathbb{R}^n) : \quad f$ is anti-symmetric in n variables$\}$

Symmetric/Boson Fock space

$H \longrightarrow$ Hilbert space

$H^{\vee n} \longrightarrow$ n-fold symmetric tensor product of H

Symmetric/Boson Fock space over H:

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} H^{\vee n}$$

where $H^{\vee 0} := \mathbb{C}$

Exponential vectors:

For $u \in H$, the exponential vector of u denoted by $e(u)$ is

$$e(u) := 1 \oplus u \oplus \frac{1}{\sqrt{2!}} u^{\otimes 2} \oplus \frac{1}{\sqrt{3!}} u^{\otimes 3} \cdots \oplus \frac{1}{\sqrt{n!}} u^{\otimes n} \oplus \cdots$$

The set of exponential vectors is a total subset of $< mathcalF(H)$

Fock space and probability

Standard Gaussian distribution on $\mathbb{R}$:

$$\mathcal{F}(\mathbb{C}) \cong L^2(\mathbb{R} , \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx)$$

Given by the unitary:

$$U(e_n) = \frac{1}{\sqrt{n!}} H_n(x)$$

where $(e_n)_n$ is the standard basis for $\mathcal{F}(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus \cdots$ and H_n is the n-th Hermite polynomial on $\mathbb{R}$

Poisson distribution with mean λ on $\mathbb{Z}_+$:

$$\mathcal{F}(\mathbb{C}) \cong L^2 \left(\mathbb{Z}_+ , \left(p(k) = \frac{\lambda^k e^{-\lambda}}{k!} \right)_{k \in \mathbb{Z}_+} \right)$$

Given by the unitary:

$$U(e_n) = \frac{1}{\sqrt{n!}} \pi_n(\lambda, x), \quad x \in \mathbb{Z}_+$$

where $\pi_n(\lambda, x)$ is the n-th Charlier-Poisson polynomial on $\mathbb{Z}_+$

Fock space and probability: contd.

Let $\{W_t : (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}\}_{t\geq 0}$ be the standard Brownian motion i.e.

- $W_0 = 0$
- For $0 < t_1 < t_2 < \cdots t_n < \infty$, $\{W_{t_i} - W_{t_{i-1}}\}_{i=1}^{n}$ are independent random variables
- $\mathbb{P}\{\omega \in \Omega : \; W_t(\omega) - W_s(\omega) \in [a, b]\ \} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b \, e^{-\frac{x^2}{(t-s)}} dx$

Wiener-Itô isomorphism :

$$\mathcal{F}\left(L^2(\mathbb{R}_+)\right) \cong L^2(\Omega, \mathbb{P})$$

Given by the unitary:

$$[Ue(f)]\,(\omega) = e^{\int_0^\infty f(t)\; dW_t(\omega) - \frac{1}{2}\int_0^\infty f(t)^2 dt} \quad (f \in L^2(\mathbb{R}_+))$$

where $\int_0^\infty f(t)\; dW_t(\omega)$ is the stochastic integral w.r.t. Brownian motion

Annihilation, Creation and Preservation

$H \longrightarrow$ Hilbert space $\qquad \mathcal{F}(H) \longrightarrow$ Symmetric Fock space on H

Annihilation:

For $u \in H$, annihilation with u is defined as

$$a(u)\,(e(\xi)) := \langle u, \xi \rangle e(\xi) \quad (\forall\ \xi \in H)$$

Creation

For $u \in H$, creation with u is defined as

$$a^\dagger(u)\,(e(\xi)) := a(u)^*\,(e(\xi))$$

i.e.

$$\langle a^\dagger(u) e(\xi), e(\eta) \rangle = \langle e(\xi), a(u) e(\eta) \rangle \quad (\xi, \eta \in H)$$

Preservation/Number:

For $u \in H$, it is defined as:

$$\Lambda\,(|u >< u|)\,(e(\xi)) := \langle u, \xi \rangle a^\dagger(u)(e(\xi)) \quad (\xi \in H)$$

Brownian motion: Annihilation + Creation

$\{W_t : (\Omega, \mathbb{P}) \longrightarrow \mathbb{R}\}_{t \geq 0}$: Standard Brownian motion

$$\mathcal{F}\left(L^2(\mathbb{R}_+)\right) \xrightarrow[\text{Wiener-Itô isomorphism}]{\text{U}} L^2(\Omega, \mathbb{P})$$

Fock space realization of Brownian motion:

$$U^* \cdot W_t \cdot U = a\left(1_{[0,t]}\right) + a^\dagger\left(1_{[0,t]}\right)$$

Hopf* algebra:

Noncommutative way of looking at compact groups

Hopf* algebra: (A, Δ, ϵ, S)

- $A \longrightarrow$ Unital *-algebra
- $\Delta : A \longrightarrow A \odot A$: unital *-homomorphism satisfying:
$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$$
- $\epsilon : A \longrightarrow \mathbb{C}$: unital *-homomorphism satisfying:
$$(\epsilon \otimes \iota)(\Delta(x)) = (\iota \otimes \epsilon)(\Delta(x)) = x \ \forall \ x \in A$$
- $S : A \longrightarrow A$: unital anti-homomorphism satisfying:
$$m \circ (S \otimes \iota) \circ \Delta = m \circ (\iota \otimes S) \circ \Delta = \epsilon$$
where m is the multiplication

Compact groups as Hopf* algebras

From compact group to Hopf* algebra

G: compact group

$Pol(G)$: the trigonometric polynomial algebra of G

Define:

- $\Delta : Pol(G) :\longrightarrow Pol(G) \odot Pol(G)$: $\Delta(\pi_{ij}) := \sum_{k=1}^{n} \pi_{ik} \otimes \pi_{kj}$
- $\epsilon : Pol(G) \longrightarrow \mathbb{C}$: $\epsilon(\pi_{ij}) := \delta_{ij}$
- $S : Pol(G) \longrightarrow Pol(G)$: $S(\pi_{ij}) := (\pi^c)_{ij}$
  (π^c: contragredient rep. of π)

Then $(Pol(G), \Delta, \epsilon, S)$ is a Hopf* algebra

From Hopf* algebra to compact group

(A, Δ, ϵ, S) : Hopf* algebra

If A is commutative then there exists a compact group G such that

- $A = Pol(G)$
- Δ, ϵ, S are like above

Group algebra of discrete group as Hopf* algebra

Γ : Discrete group
$\mathbb{C}\Gamma := \text{Lin}\{\lambda\delta_g : \lambda \in \mathbb{C}, g \in \Gamma\}$ (Group algebra of Γ)

$\mathbb{C}\Gamma$ is a Hopf* algebra:

Define:

- $\Delta : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \odot \mathbb{C}\Gamma \,:\, \Delta(\delta_g) := \delta_g \otimes \delta_g$
- $\epsilon : \mathbb{C}\Gamma \longrightarrow \mathbb{C} \,:\, \epsilon(\delta_g) := 1$
- $S : \mathbb{C}\Gamma \longrightarrow \mathbb{C}\Gamma \,:\, S(\delta_g) := \delta_{g^{-1}}$

Then $(\mathbb{C}\Gamma, \Delta, \epsilon, S)$ is a Hopf* algebra

Convolution in Hopf* algebra

(A, Δ, ϵ, S) : Hopf* algebra B : *-algebra

$P : A \longrightarrow B$ & $Q : A \longrightarrow B$: linear maps

Convolution of P & Q denoted by $P \star Q : A \longrightarrow B$:

For $x \in A$ let $\Delta(x) = \sum_{i=1}^{n} x_i \otimes y_i$.

Then

$$(P \star Q)[x] := \sum_{i=1}^{n} P(x_i)Q(y_i)$$

i.e. $P \star Q = m_B \circ (P \otimes Q) \circ \Delta$, where m_B is the multiplication in B

Quantum Lévy process on a Hopf* algebra

Ingredients for definning a quantum Lévy process

(A, Δ, ϵ, S) : a Hopf* algebra   k : Hilbert space (noise space)

$B(\mathsf{k})$: set of bounded operators on k

ϵ-ρ cocycle on (A, Δ, ϵ, S) with noise k : $\eta : A \longrightarrow \mathsf{k}$

Let $\rho : A \longrightarrow B(\mathsf{k})$ be a unital *-homomorphism.

A linear map $\eta : A \longrightarrow \mathsf{k}$ is called ϵ-ρ cocycle if

$$\eta(xy) = \eta(x)\epsilon(y) + \rho(x)\,(\eta(y)) \quad (\forall\ x, y \in A)$$

e.g.

For any *-homomorphism $\pi : A \longrightarrow B(\mathsf{k})$ and $\xi \in \mathsf{k}$, define
$\eta : A \longrightarrow \mathsf{k}$ by $\eta(x) := \pi(x)\xi - \epsilon(x)\xi$.

Then $\eta : A \longrightarrow \mathsf{k}$ is a ϵ-ρ cocycle on (A, Δ, ϵ, S) with noise k

Ingredients for definning a quantum Lévy process.....contd.

(A, Δ, ϵ, S) : a Hopf* algebra k : Hilbert space (noise space)
$B(\mathrm{k})$: set of bounded operators on k

Positive functional on (A, Δ, ϵ, S) : $\phi : A \longrightarrow \mathbb{C}$

A linear map $\phi : A \longrightarrow \mathbb{C}$ is called a positive functional if:

- $\phi(x^*) = \overline{\phi(x)}$ for all $x \in A$
- $\phi(x^* x) \geq 0$ for all $x \in A$. ϕ is called a state if $\phi(1) = 1$

Conditionally positive functional on (A, Δ, ϵ, S) : $\gamma : A \longrightarrow \mathbb{C}$

A linear map $\gamma : A \longrightarrow \mathbb{C}$ is called a conditionally positive functional if:

- $\gamma(x^*) = \overline{\gamma(x)}$ for all $x \in A$
- $\gamma(x^* x) \geq 0$ whenever $\epsilon(x) = 0$ (condition on positivity)

e.g. Let $\pi : A \longrightarrow B(\mathrm{k})$ be any *-homomorphism and $\xi \in \mathrm{k}$.
Define $\phi : A \longrightarrow \mathbb{C}$ by $\phi(x) := \langle \xi, \pi(x)\xi \rangle$. ϕ is positive.
Define $\gamma : A \longrightarrow \mathbb{C}$ by $\gamma(x) := \phi(x) - \epsilon(x).1$. γ is cond. positive.

Convolution semigroup of states

(A, Δ, ϵ, S) : Hopf* algebra

Convolution semigroup of states:

A semigroup of states $\{\phi_t : A \longrightarrow \mathbb{C}\}_{t \geq 0}$ satisfying

- $\phi_0 = \epsilon$ & $\phi_t \star \phi_s = \phi_{t+s}$
- $\mathbb{R}_+ \ni t \mapsto \phi_t(x) \in \mathbb{C}$ is continuous for each $x \in A$

e.g.
Let G be a compact group and $\{\mu_t\}_{t \geq 0}$ be a convolution semigroup of probability measures on G.

Then $\{\phi_t : Pol(G) \longrightarrow \mathbb{C}\}_{t \geq 0}$ defined by $\phi_t(\pi_{ij}) := \int_G \pi_{ij}(g) d\mu_t(g)$ is a convolution semigroup of states on the Hopf* algebra $(Pol(G), \Delta, \epsilon, S)$.

Schoenberg correspondence

(A, Δ, ϵ, S) : Hopf* algebra

From semigroup to conditionally positive functional:

Let $\{\phi_t : A \longrightarrow \mathbb{C}\}_{t \geq 0}$ be a convolution semigroup of states.
Then

$$\gamma(x) := \frac{d}{dt}|_{t=0} \phi_t(x) \quad (x \in A)$$

defines a conditionally positive functional.

From conditionally positive functional to semigroup:

Let $\gamma : A \longrightarrow \mathbb{C}$ be a conditionally positive functional.
Then there exists a convolution semigroup of states
$\{\phi_t : A \longrightarrow \mathbb{C}\}_{t \geq 0}$ such that

$$\gamma(x) = \frac{d}{dt}|_{t=0} \phi_t(x) \quad (x \in A).$$

Quantum Lévy process

(A, Δ, ϵ, S) : a Hopf* algebra

$\mathcal{F} := \mathcal{F}\left(L^2(\mathbb{R}_+, \mathsf{k})\right)$

k : Hilbert space (noise space)

$\Omega \in \mathcal{F}$: the vacuum vector

Definition:

A family $\{\ell_{[s,t)} : A \longrightarrow B(\mathcal{F})\}_{s<t\in\mathbb{R}_+}$ of unital *-homomorphisms satisfying:

- $\ell_{tt} = \epsilon$ & for $t_1 < t_2 < t_3$ $\ell_{[t_1,t_2)} \star \ell_{[t_2,t_3)} = \ell_{[t_1,t_3)}$
- There exists a convolution semigroup of states $\{\phi_t : A \longrightarrow \mathbb{C}\}_{t\geq 0}$ such that
  - (a) $\phi_t(x) = \langle \ell_{[a,b)}(x)\Omega, \Omega\rangle$ whenever $b - a = t$
  - (b) For $t_1 < t_2 < \cdots t_n$, $x_1, x_2, \cdots x_{n-1} \in A$

$$\left\langle \left[\prod_{i=1}^{n-1} \ell_{[t_i,t_{i+1})}(x_i)\right] \Omega, \Omega \right\rangle = \prod_{i=1}^{n-1} \phi_{t_{i+1}-t_i}(x_i)$$

Why quantum Lévy process?

Consider the Hopf* algebra $(Pol(G), \Delta, \epsilon, S)$ for a compact group G.

From quantum Lévy procecss to classical Lévy process

Let $\{\ell_{[s,t)} : Pol(G) \longrightarrow B(\mathcal{F})\}_{s<t\in\mathbb{R}_+}$ be a quantum Lévy process with associated convolution semigroup of states $\{\phi_t : Pol(G) \longrightarrow \mathbb{C}\}_{t\geq 0}$.

Then there exists a Lévy process $\{X_t : (\Sigma, \mathbb{P}) \longrightarrow G\}_{t\geq 0}$ with its associated convolution semigroup $\{\mu_t\}_{t\geq 0}$ of probability measures on Gsuch that

$$\phi_t(\pi_{ij}) = \int_G \pi_{ij}(g) d\mu_t(g) \quad (\pi_{ij} \in Pol(G)).$$

Dynamical description of quantum Lévy process

Schürmann triples: (γ, η, H)

(A, Δ, ϵ, S) : Hopf* algebra

$\gamma : A \longrightarrow \mathbb{C}$: conditionally positive functional

Schürmann triple:

There exists

- H : a Hilbert space
- $\rho : A \longrightarrow B(H)$: a unital *-homomorphism
- $\eta : A \longrightarrow H$: a ϵ-ρ cocycle

satisfying:

$$\langle \eta(x), \eta(y) \rangle = \gamma(x^* y) - \gamma(x^*)\epsilon(y) - \epsilon(x^*)\gamma(y) \quad (x, y \in A).$$

(γ, η, H) is called Schürmann triple.

Structure of quantum Lévy process

(A, Δ, ϵ, S) : Hopf* algebra

$\gamma : A \longrightarrow \mathbb{C}$: conditionally positive functional

(γ, η, H) : associated Schürmann triple

$\mathcal{F} := \mathcal{F}(L^2(\mathbb{R}_+, H))$

The driving equation: M. Schürmann, 1993

There exists a quantum Lévy process $\{\ell_{[s,t)} : A \longrightarrow B(\mathcal{F})\}_{s<t\in\mathbb{R}_+}$ satisfying:

$$\ell_{[s,t)} = \epsilon + \int_{\tau=s}^{\tau=t} \ell_{[s,\tau)} \star \Big(a_\eta(d\tau) + a_\eta^\dagger(d\tau) + \Lambda_\rho(d\tau) + \gamma d\tau \Big)$$

γ is called a Gaussian generator (or equivalently the Lévy process is called Gaussian) if $\rho = \epsilon$.

Thank you for your attention