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Fluctuations for particle systems in one dimension
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Abstract. These notes discuss limit distributions and variance bounds for particle current in several systems of particles on the one-dimensional integer lattice: independent random walks, independent random walks in a random environment, the random average process, the asymmetric simple exclusion process, and a class of zero range processes. The text is based on collaborations with Márton Balázs, Mathew Joseph, Júlia Komjáthy, Rohini Kumar, Jon Peterson and Firas Rassoul-Agha.

Contents

CHAPTER 1

Introduction

These lecture notes discuss the process of particle current in several systems of particles on the one-dimensional integer lattice. The processes studied fall into two categories determined by the slope of the flux function H: processes with a linear flux and those with a strictly concave flux. By definition, the flux $H(\rho)$ is the mean rate at which particle mass moves past a fixed point in space when the system is stationary and has overall density ρ of particles. The processes we study are asymmetric in the sense that the particles have a preferred average direction. This assumption is not necessary for the systems with linear flux, but for those with nonlinear flux it is crucial.

The examples of systems with linear flux we cover are independent random walks, independent random walks in a random environment, and the random average process. For these processes the order of magnitude of the fluctuations of the current is $n^{1/4}$ in terms of a scaling parameter n that gives both the space and time scale, and goes to ∞. The results we give describe the Gaussian limit distributions of the current process.

The systems with concave flux that we discuss are the asymmetric simple exclusion process (ASEP) and a class of zero range processes. In this case the assumption of asymmetry is necessary. We consider only stationary systems (or small perturbations thereof), and instead of distributional limits we give only bounds on the variance of the current and on the moments of a second class particle. The current fluctuations are now of order $t^{1/3}$. This order of magnitude goes together with superdiffusivity of the second class particle whose fluctuations are of order $t^{2/3}$. In statistical physics terminology, these systems are in the KPZ class (Kardar-Parisi-Zhang). Here t is the time parameter of the process. A separate scaling parameter is not needed since we have no process-level result.

We cannot cover complete proofs for all results in this short lecture series. The most important result, namely the moment bounds for the second-class particle in ASEP, is proved in full detail, assuming some basic facts about ASEP.

The most glaring omission is that we do not treat the Tracy-Widom fluctuations of TASEP. This topic would require a lecture series of its own. In the ASEP chapter we state briefly the Ferrari-Spohn theorem on the limit distribution of the current across a characteristic.

The exercises at the end of chapters are intended for students with some background in probability but not necessarily in the issues relevant to these notes.

Notation. $\mathbb{Z}_+ = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} = \{1, 2, 3, \dots\}$.

CHAPTER 2

Independent particles executing classical random walks

2.1. Model and results

Fix a probability distribution $p(x)$ on $\mathbb{Z}$. Let us assume for simplicity that $p(x)$ has finite range, that is,

$$\text{the set } \{x \in \mathbb{Z} : p(x) > 0\} \text{ is finite.} \tag{2.1}$$

For each site $x \in \mathbb{Z}$ and index $k \in \mathbb{N}$ let $X_{\centerdot}^{x,k}$ be a discrete-time random walk with initial point $X_0^{x,k} = x$ and transition probability

$$P(X_{s+1}^{x,k} + y \,|\, X_s^{x,k} = x) = p(y-x) \quad \text{for times } s \in \mathbb{Z}_+ \text{ and space points } x, y \in \mathbb{Z}.$$

The walks $\{X_{\centerdot}^{x,k} : x \in \mathbb{Z}, k \geq 1\}$ are independent of each other. Let

$$v = \sum_{x \in \mathbb{Z}} x p(x) \quad \text{and} \quad \sigma_1^2 = \sum_{x \in \mathbb{Z}} (x - v)^2 p(x)$$

be the mean speed and variance of the walks.

At time 0 we start a random number $\eta_x(0)$ of particles at site x. The assumption is that the initial occupation variables $\eta(0) = \{\eta_x(0) : x \in \mathbb{Z}\}$ are i.i.d. (independent and identically distributed) with finite mean and variance

$$\mu_0 = E[\eta_x(0)] \quad \text{and} \quad \sigma_0^2 = \mathrm{Var}[\eta_x(0)]. \tag{2.2}$$

Furthermore, the variables $\{\eta_x(0)\}$ and the walks $\{X_{\centerdot}^{x,k}\}$ are independent of each other. If the locations of individual particles are not of interest but only the overall particle distribution, the particle configuration at time $s \in \mathbb{N}$ is described by the occupation variables $\eta(s) = \{\eta_x(s) : x \in \mathbb{Z}\}$ defined as

$$\eta_x(s) = \sum_{y \in \mathbb{Z}} \sum_{k=1}^{\eta_0(y)} \mathbf{1}\{X_s^{y,k} = x\}.$$

Let $n \in \mathbb{N}$ denote a scaling parameter that eventually goes to ∞. For (macroscopic) times $t \in \mathbb{R}_+$ and a spatial variable $r \in \mathbb{R}$, let

$$Y_n(t,r) = Y_{n,1}(t,r) - Y_{n,2}(t,r) \tag{2.3}$$

with

$$Y_{n,1}(t,r) = \sum_{x \leq 0} \sum_{k=1}^{\eta_x(0)} \mathbf{1}\{X_{\lfloor nt \rfloor}^{x,k} > \lfloor ntv \rfloor + r\sqrt{n}\,\} \tag{2.4}$$

and

$$Y_{n,2}(t,r) = \sum_{x > 0} \sum_{k=1}^{\eta_x(0)} \mathbf{1}\{X_{\lfloor nt \rfloor}^{x,k} \leq \lfloor ntv \rfloor + r\sqrt{n}\,\}. \tag{2.5}$$

Variable $Y_n(t,r)$ represents the net left-to-right current of particles seen by an observer who starts at the origin and reaches point $\lfloor ntv\rfloor + \lfloor r\sqrt{n}\rfloor$ in time $\lfloor nt\rfloor$. Its mean is

$$EY_n(t,r) = \mu_0 E\big(X_{\lfloor nt\rfloor} - \lfloor ntv\rfloor - \lfloor r\sqrt{n}\,\rfloor\big) = -\mu_0 r\sqrt{n} + O(1)$$

Define the centered and appropriately scaled process by

$$\overline{Y}_n(t,r) = n^{-1/4}\big(Y_n(t,r) - EY_n(t,r)\big).$$

The goal is to prove a limit for the joint distributions of these random variables. We will not tackle process-level convergence. But let us point out that there is a natural path space D_2 of functions of two parameters (t,r) that contains the paths of the processes Y_n. Elements of D_2 are continuous from above with limits from below in a suitable way, and there is a metric that generalizes the standard Skorohod topology of the usual D-space of cadlag paths. (See [**BW71, Kum08**].)

Let

$$\varphi_{\nu^2}(x) = \frac{1}{\sqrt{2\pi\nu^2}}\exp\Big(-\frac{x^2}{2\nu^2}\Big) \quad\text{and}\quad \Phi_{\nu^2}(x) = \int_{-\infty}^{x}\varphi_{\nu^2}(y)dy \tag{2.6}$$

denote the centered Gaussian density with variance ν^2 and its distribution function. Let W be a two-parameter Brownian motion on $\mathbb{R}_+\times\mathbb{R}$ and B a two-sided one-parameter Browian motion on $\mathbb{R}$. W and B are independent. Define the process Z by

$$\begin{aligned} Z(t,r) = \sqrt{\mu_0}\iint_{[0,t]\times\mathbb{R}} &\varphi_{\sigma_1^2(t-s)}(r-x)\,dW(s,x) \\ &+ \sigma_0\int_{\mathbb{R}}\varphi_{\sigma_1^2 t}(r-x)B(x)\,dx. \end{aligned} \tag{2.7}$$

$\{Z(t,r) : t\in\mathbb{R}_+, r\in\mathbb{R}\}$ is a mean zero Gaussian process. Its covariance can be expressed as follows: with

$$\Psi_{\nu^2}(x) = \nu^2\varphi_{\nu^2}(x) - x\big(1-\Phi_{\nu^2}(x)\big) \tag{2.8}$$

define

$$\Gamma_1\big((s,q),(t,r)\big) = \Psi_{\sigma_1^2(t+s)}(r-q) - \Psi_{\sigma_1^2|t-s|}(r-q) \tag{2.9}$$

and

$$\Gamma_2\big((s,q),(t,r)\big) = \Psi_{\sigma_1^2 s}(-q) + \Psi_{\sigma_1^2 t}(r) - \Psi_{\sigma_1^2(t+s)}(r-q). \tag{2.10}$$

Then

$$\mathbf{E}[Z(s,q)Z(t,r)] = \mu_0\Gamma_1\big((s,q),(t,r)\big) + \sigma_0^2\Gamma_2\big((s,q),(t,r)\big). \tag{2.11}$$

This is verified in the exercises.

It is also the case that the process $Z(t,r)$ is the unique weak solution of the following initial value problem for a stochastic heat equation on $\mathbb{R}_+\times\mathbb{R}$:

$$Z_t = \tfrac{\sigma_1^2}{2}Z_{rr} + \sqrt{\mu_0}\,\dot{W}\,, \quad Z(0,r) = \sigma_0 B(r). \tag{2.12}$$

(Above, subscript means partial derivative.) A weak solution of this equation is defined by the requirement

$$\begin{aligned} &\int_{\mathbb{R}}\phi(r)Z(t,r)\,dr - \sigma_0\int_{\mathbb{R}}\phi(r)B(r)\,dr \\ &\quad = \tfrac{\sigma_1^2}{2}\iint_{[0,t]\times\mathbb{R}}\phi''(r)Z(s,r)\,dr\,ds + \sqrt{\mu}_0\iint_{[0,t]\times\mathbb{R}}\phi(r)dW(s,r) \end{aligned} \tag{2.13}$$

for all $\phi\in C_c^\infty(\mathbb{R})$ (compactly supported, infinitely differentiable). See the lecture notes of Walsh [**Wal86**].

THEOREM 2.1. *Assume the initial occupation variables are i.i.d. with finite mean and variance as in* (2.2). *Then as* $n \to \infty$, *the finite-dimensional distributions of the process* $\{\overline{Y}_n(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$ *converge weakly to the finite-dimensional distributions of the mean zero Gaussian process* $\{Z(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$.

The statement means that for any space-time points $(t_1, r_1), \dots, (t_k, r_k)$, this weak convergence of $\mathbb{R}^k$-valued random vectors holds:

$$(\overline{Y}_n(t_1,r_1), \dots, \overline{Y}_n(t_k,r_k)) \xrightarrow{\mathcal{D}} (Z(t_1,r_1), \dots, Z(t_k,r_k)).$$

Under additional moment assumptions process level convergence in the space D_2 can be proved (see [**Kum08**]). We state a corollary for the special case of the stationary occupation process $\eta(t)$.

COROLLARY 2.2. *Under the invariant distribution where* $\{\eta_x(t) : x \in \mathbb{Z}\}$ *are i.i.d. Poisson with mean* μ_0 *for each fixed* t, *at* $r = 0$ *the limit process* Z *has covariance*

$$\mathbf{E}Z(s,0)Z(t,0) = \frac{\mu_0 \sigma_1}{\sqrt{2\pi}} \big(\sqrt{s} + \sqrt{t} - \sqrt{|t-s|} \big),$$

i.e., process $Z(\cdot, 0)$ *is fractional Brownian motion with Hurst parameter* $1/4$.

2.2. Sketch of proof

We turn to discuss the proof. Independent walks allow us to compute many things in a straightforward manner. Fix some $N \in \mathbb{N}$, time points $0 < t_1 < t_2 < \cdots < t_N \in \mathbb{R}_+$, space points $r_1, r_2, \dots, r_N \in \mathbb{R}$ and an N-vector $\boldsymbol{\theta} = (\theta_1, \dots, \theta_N) \in \mathbb{R}^N$. Form the linear combinations

$$\overline{Y}_n(\boldsymbol{\theta}) = \sum_{i=1}^N \theta_i \overline{Y}_n(t_i, r_i) \quad \text{and} \quad Z(\boldsymbol{\theta}) = \sum_{i=1}^N \theta_i Z(t_i, r_i).$$

The goal is now to prove

PROPOSITION 2.3.

$$E\big[\exp\{i\overline{Y}_n(\boldsymbol{\theta})\}\big] \to \mathbf{E}\big[\exp\{iZ(\boldsymbol{\theta})\}\big]. \tag{2.14}$$

Since the random walks and initial occupation variables are independent, we can write $\overline{Y}_n(\boldsymbol{\theta})$ as a sum of independent random variables and take advantage of standard central limit theorems from the literature.

$$\overline{Y}_n(\boldsymbol{\theta}) = n^{-\frac{1}{4}} \sum_{i=1}^N \theta_i \big\{ Y_n(t_i, r_i) - EY_n(t_i, r_i) \big\} = W_n = \sum_{m=-\infty}^{\infty} u(m) \tag{2.15}$$

with

$$u(m) = \sum_{i=1}^N \theta_i \Big(U_m(t_i, r_i) \, \mathbf{1}\{m \le 0\} - V_m(t_i, r_i) \, \mathbf{1}\{m > 0\} \Big), \tag{2.16}$$

and

$$U_m(t,r) = n^{-\frac{1}{4}} \sum_{j=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,j} > \lfloor ntv \rfloor + r\sqrt{n}\} - n^{-\frac{1}{4}} \mu_0 P(X_{nt}^m > \lfloor ntv \rfloor + r\sqrt{n}\,), \tag{2.17}$$

$$V_m(t,r) = n^{-\frac{1}{4}} \sum_{j=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,j} \le \lfloor ntv \rfloor + r\sqrt{n}\} - n^{-\frac{1}{4}} \mu_0 P(X_{nt}^m \le \lfloor ntv \rfloor + r\sqrt{n}\,).$$

The variables $\{u(m)\}_{m\in\mathbb{Z}}$ are independent because initial occupation variables and walks are independent.

Let $a(n) \nearrow \infty$ be a sequence that will be determined precisely in the proof. Define

$$W_n^* = \sum_{|m|\le a(n)\sqrt{n}} u(m). \tag{2.18}$$

As an exercise, check that the terms $|m| > a(n)\sqrt{n}$ can be discarded from (2.15):

Lemma 2.4. *$E|W_n - W_n^*|^2 \to 0$ as $n \to \infty$.*

The limit $Z(\boldsymbol{\theta})$ in our goal (2.14) has $\mathcal{N}(0, \sigma(\boldsymbol{\theta})^2)$ distribution with variance

$$\sigma(\boldsymbol{\theta})^2 = \sum_{1\le i,j\le N} \theta_i\theta_j \Big[\mu_0\Gamma_1\big((t_i,r_i),(t_j,r_j)\big) + \sigma_0^2\Gamma_2\big((t_i,r_i),(t_j,r_j)\big)\Big]. \tag{2.19}$$

The two Γ-terms, defined earlier in (2.9) and (2.10), have the following expressions in terms of a standard 1-dimensional Brownian motion B_t:

$$\Gamma_1\big((s,q),(t,r)\big) = \int_{-\infty}^{\infty} \Big(\mathbf{P}[B_{\sigma_1^2 s} \le q-x]\mathbf{P}[B_{\sigma_1^2 t} > r-x] - \mathbf{P}[B_{\sigma_1^2 s} \le q-x, B_{\sigma_1^2 t} > r-x]\Big)\,dx \tag{2.20}$$

and

$$\Gamma_2\big((s,q),(t,r)\big) = \int_{-\infty}^{0} \mathbf{P}[B_{\sigma_1^2 s} > q-x]\mathbf{P}[B_{\sigma_1^2 t} > r-x]\,dx + \int_0^{\infty} \mathbf{P}[B_{\sigma_1^2 s} \le q-x]\mathbf{P}[B_{\sigma_1^2 t} \le r-x]\,dx. \tag{2.21}$$

By Lemma 2.4, the desired limit (2.14) follows from showing

$$E(e^{iW_n^*}) \to e^{-\sigma(\boldsymbol{\theta})^2/2}. \tag{2.22}$$

This will be achieved by the Lindeberg-Feller theorem.

Theorem 2.5 (Lindeberg-Feller). *For each n, suppose $\{X_{n,j} : 1 \le j \le J(n)\}$ are independent, mean-zero, square-integrable random variables and let $S_n = X_{n,1} + \cdots + X_{n,J(n)}$. Assume that*

$$\lim_{n\to\infty} \sum_{j=1}^{J(n)} E(X_{n,j}^2) = \sigma^2$$

and for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{J(n)} E\big(X_{n,j}^2 \mathbf{1}\{|X_{n,j}| \ge \varepsilon\}\big) = 0.$$

Then as $n \to \infty$, S_n converges in distribution to a $\mathcal{N}(0, \sigma^2)$-distributed Gaussian random variable. In terms of probabilities, the conclusion is that for all $s \in \mathbb{R}$,

$$\lim_{n\to\infty} P\{S_n \le s\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{s} e^{-x^2/2\sigma^2}\, dx.$$

In terms of characteristic functions, the conclusion is that for all $t \in \mathbb{R}$,

$$\lim_{n\to\infty} E(e^{itS_n}) = e^{-\sigma^2 t^2/2}.$$

Now to prove (2.22) the task is to verify the conditions of the Lindeberg-Feller theorem:

$$\sum_{|m|\le a(n)\sqrt{n}} E(u(m)^2) \to \sigma(\boldsymbol{\theta})^2 \tag{2.23}$$

and

$$\sum_{|m|\le a(n)\sqrt{n}} E\big(\,|u(m)|^2 \mathbf{1}\{|u(m)| \ge \varepsilon\}\big) \to 0. \tag{2.24}$$

We begin with the negligibility condition (2.24). This will determine $a(n)$.

Lemma 2.6. *Under assumption* (2.2),

$$\lim_{n\to\infty} \sum_{|m|\le a(n)\sqrt{n}} E\big(\,|u(m)|^2 \mathbf{1}\{|u(m)| \ge \varepsilon\}\big) = 0. \tag{2.25}$$

Proof. Since

$$|u(m)| \le Cn^{-1/4}\big(\eta_m(0) + \mu_0\big)$$

so for a different $\varepsilon_1 > 0$ and by shift-invariance,

$$\sum_{|m|\le a(n)\sqrt{n}} E\big[|u(m)|^2 \mathbf{1}\{|u(m)| \ge \varepsilon\}\big] \le Ca(n)E\big[(\eta_0(0) + \mu_0)^2 \mathbf{1}\{\eta_0(0) \ge n^{1/4}\varepsilon_1\}\big].$$

By the moment assumption (2.2) this last expression $\to 0$ for every $\varepsilon_1 > 0$ if $a(n) \nearrow \infty$ slowly enough, for example

$$a(n) = \Big(E\big[(\eta_0(0) + \mu_0)^2 \mathbf{1}\{\eta_0(0) \ge n^{1/8}\}\big]\Big)^{-1/2}$$

□

We turn to checking (2.23).

$$\begin{aligned}
&\sum_{|m|\le a(n)\sqrt{n}} E\big[u(m)^2\big] \\
&= \sum_{1\le i,j\le N} \theta_i\theta_j \sum_{|m|\le a(n)\sqrt{n}} \Big[\mathbf{1}_{\{m\le 0\}} E\big(U_m(t_i, r_i)U_m(t_j, r_j)\big) \\
&\qquad\qquad + \mathbf{1}_{\{m>0\}} E\big(V_m(t_i, r_i)V_m(t_j, r_j)\big)\Big].
\end{aligned} \tag{2.26}$$

To the expectations we apply this formula for the covariance of two random sums: with $\{Z_i\}$ i.i.d. and independent of a random $K \in \mathbb{Z}_+$,

$$\mathrm{Cov}\Big(\sum_{i=1}^{K} f(Z_i), \sum_{j=1}^{K} g(Z_j)\Big) = EK\ \mathrm{Cov}(f(Z), g(Z)) + \mathrm{Var}(K)\, Ef(Z)\, Eg(Z). \tag{2.27}$$

For the first expectation on the right in (2.26):

$$
\begin{aligned}
&E\big(U_m(s,q)U_m(t,r)\big)\\
&= n^{-1/2}\operatorname{Cov}\Big(\sum_{j=1}^{\eta_m(0)} \mathbf{1}\{X^{m,j}_{ns} > \lfloor nsv\rfloor + q\sqrt{n}\,\},\ \sum_{j=1}^{\eta_m(0)} \mathbf{1}\{X^{m,j}_{nt} > \lfloor ntv\rfloor + r\sqrt{n}\,\}\Big)\\
&= n^{-1/2}\mu_0\Big[P(X^m_{ns} > \lfloor nsv\rfloor + q\sqrt{n},\, X^m_{nt} > \lfloor ntv\rfloor + r\sqrt{n}\,)\\
&\qquad\qquad -\ P(X^m_{ns} > \lfloor nsv\rfloor + q\sqrt{n}\,)P(X^m_{nt} > \lfloor ntv\rfloor + r\sqrt{n}\,)\Big]\\
&\quad + n^{-1/2}\sigma_0^2 P(X^m_{ns} > \lfloor nsv\rfloor + q\sqrt{n}\,)P(X^m_{nt} > \lfloor ntv\rfloor + r\sqrt{n}\,).
\end{aligned}
$$

Do the same for the V-terms. After some rearranging of the probabilities, we arrive at

$$
\begin{aligned}
&\sum_{|m|\le a(n)\sqrt{n}} E\big[u(m)^2\big]\\
&= n^{-1/2}\sum_{1\le i,j\le N}\theta_i\theta_j\Big[\mu_0\sum_{|m|\le a(n)\sqrt{n}}\Big\{P(X^m_{nt_i}\le \lfloor nt_iv\rfloor + r_i\sqrt{n}\,)P(X^m_{nt_j} > \lfloor nt_jv\rfloor + r_j\sqrt{n}\,)\\
&\qquad\qquad -\ P(X^m_{nt_i}\le \lfloor nt_iv\rfloor + r_i\sqrt{n},\, X^m_{nt_j} > \lfloor nt_jv\rfloor + r_j\sqrt{n}\,)\Big\}\\
&\quad + \sigma_0^2\sum_{-a(n)\sqrt{n}\le m\le 0} P(X^m_{nt_i} > \lfloor nt_iv\rfloor + r_i\sqrt{n}\,)P(X^m_{nt_j} > \lfloor nt_jv\rfloor + r_j\sqrt{n}\,)\\
&\quad + \sigma_0^2\sum_{0< m\le a(n)\sqrt{n}} P(X^m_{nt_i}\le \lfloor nt_iv\rfloor + r_i\sqrt{n}\,)P(X^m_{nt_j}\le \lfloor nt_jv\rfloor + r_j\sqrt{n}\,)\Big].
\end{aligned}
\tag{2.28}
$$

The terms above have been arranged so that the sums match up with the integrals in (2.19)–(2.21). Limit (2.23) now follows because each sum converges to the corresponding integral. To illustrate with the last term, the convergence needed is

$$
\begin{aligned}
&n^{-1/2}\sum_{0<m\le a(n)\sqrt{n}} P(X^m_{ns}\le \lfloor nsv\rfloor + q\sqrt{n}\,)P(X^m_{nt}\le \lfloor ntv\rfloor + r\sqrt{n}\,)\\
&= n^{-1/2}\sum_{0<m\le a(n)\sqrt{n}} P\Big\{\frac{X_{ns}-\lfloor nsv\rfloor}{\sqrt{n}}\le q-\frac{m}{\sqrt{n}}\Big\}P\Big\{\frac{X_{nt}-\lfloor ntv\rfloor}{\sqrt{n}}\le r-\frac{m}{\sqrt{n}}\Big\}\\
&\underset{n\to\infty}{\longrightarrow}\int_0^\infty \mathbf{P}[B_{\sigma_1^2 s}\le q-x]\mathbf{P}[B_{\sigma_1^2 t}\le r-x]\,dx.
\end{aligned}
\tag{2.29}
$$

This follows from the CLT, a Riemann sum type argument and some estimation. We skip the details. With this we consider Theorem 2.1 proved.

Problems

Exercise 2.1. (a) If you have never done so, prove that i.i.d. Poisson occupations are invariant for independent random walks.

(b) Prove some version of the statement that, if initial occupation variables are i.i.d. (or merely ergodic) with common mean μ, then as $t\to\infty$ the distribution of η_t converges to i.i.d. Poissons with mean μ.

EXERCISE 2.2. Compute the flux function as the expected rate of flow:

$$H(\mu) = E^{\mu}[\text{net number of particles that jump across edge } (0,1) \text{ left to right in one time step}]. \tag{2.30}$$

Here E^{μ} denotes expectation under the stationary process whose initial occupation variables are i.i.d. Poisson with common mean μ. You should find the linear flux $H(\mu) = v\mu$.

EXERCISE 2.3. In this exercise you prove a very simple case of a hydrodynamic limit, namely for independent particles.

(a) Let ρ_0 be a bounded, differentiable function on $\mathbb{R}$ such that $0 \le \rho_0(x) \le C$ for some constant $C < \infty$. Observe that $\rho(t,x) = \rho_0(x - tv)$ solves the initial value problem

$$\rho_t + H(\rho)_x = 0\,, \quad \rho(0,\cdot\,) = \rho_0(\cdot).$$

(b) Consider a sequence of processes of independent random walks on $\mathbb{Z}$ indexed by $n = 1, 2, 3, \ldots$ and let $\eta^n(t) = \{\eta^n_x(t) : x \in \mathbb{Z}\}$ denote the occupation variables of process $\eta^n(\cdot)$. Assume that for each n the initial occupation variables $\{\eta^n_x(0) : x \in \mathbb{Z}\}$ are independent with means $E[\eta^n_x(0)] = \rho_0(x/n)$ for $x \in \mathbb{Z}$, and variances are uniformly bounded: $\mathrm{Var}[\eta^n_x(0)] \le C$ for all $n \in \mathbb{N}$ and $x \in \mathbb{Z}$. Show that ρ_0 represents the macroscopic density profile at time 0, in the sense that

$$\lim_{n\to\infty} E\biggl|\frac{1}{n}\sum_{x=\lfloor na\rfloor}^{\lfloor nb\rfloor -1} \eta^n_x(0) \;-\; \int_a^b \rho_0(x)\,dx\biggr| = 0 \quad \text{for all } a < b \text{ in } \mathbb{R}. \tag{2.31}$$

(c) Show that at later (scaled) times, the process obeys the density profile $\rho(t,x)$, in the sense that

$$\lim_{n\to\infty} E\biggl|\frac{1}{n}\sum_{x=\lfloor na\rfloor}^{\lfloor nb\rfloor -1} \eta^n_x(nt) \;-\; \int_a^b \rho(t,x)\,dx\biggr| = 0 \quad \text{for all } a < b \text{ in } \mathbb{R} \text{ and } 0 < t < \infty. \tag{2.32}$$

EXERCISE 2.4. Show that if Γ_1 is defined by (2.20) then it satisfies (2.9). For this you might note that $(d/dx)\Psi_{\nu^2}(x) = -\Phi_{\nu^2}(-x)$.

Then via $(d/d\nu)\Psi_{\nu^2}(x)$ show that

$$\Gamma_1\bigl((s,q),(t,r)\bigr) = \tfrac{1}{2}\int_{\sigma_1^2|t-s|}^{\sigma_1^2(t+s)} \frac{1}{\sqrt{2\pi v}}\exp\Bigl\{\frac{1}{2v}(r-q)^2\Bigr\}\,dv. \tag{2.33}$$

EXERCISE 2.5. Define the process

$$\zeta(t,r) = \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma_1^2(t-s)}(r-x)\,dW(s,x).$$

Show that it has covariance

$$\mathbf{E}\zeta(s,q)\zeta(t,r) = \tfrac{1}{2}\int_{\sigma_1^2|t-s|}^{\sigma_1^2(t+s)} \frac{1}{\sqrt{2\pi v}}\exp\Bigl\{\frac{1}{2v}(r-q)^2\Bigr\}\,dv. \tag{2.34}$$

This is a straightforward computation but you need to know that in general for $f, g \in L^2(\mathbb{R}^d)$ the white-noise integrals $\int f\,dW$ and $\int g\,dW$ on $\mathbb{R}^d$ are by definition mean zero Gaussian random variables that satisfy

$$\mathbf{E}\Bigl[\Bigl(\int f\,dW\Bigr)\Bigl(\int g\,dW\Bigr)\Bigr] = \int_{\mathbb{R}^d} f(x)g(x)\,dx.$$

EXERCISE 2.6. Show that if Γ_2 is defined by (2.21) then it satisfies (2.10). One way to proceed is via $\Phi_{\nu^2}(x) = 1 - \Phi_{\nu^2}(-x)$ and

$$\int_{-\infty}^{\infty} \Phi_{\alpha^2}(x)\Phi_{\nu^2}(r-x)\,dx = \int_{-\infty}^{r} \Phi_{\alpha^2+\nu^2}(x)\,dx.$$

Then show that the process

$$\xi(t,r) = \int_{\mathbb{R}} \varphi_{\sigma_1^2 t}(r-x)B(x)\,dx$$

is a mean-zero Gaussian process with covariance

$$\mathbf{E}\xi(s,q)\xi(t,r) = \Gamma_2\big((s,q),(t,r)\big).$$

Here $B(x)$ is a two-sided Brownian motion which means that we take two independent standard Brownian motions B_1 and B_2 and set

$$B(x) = \begin{cases} B_1(x), & x \geq 0 \\ B_2(-x), & x < 0. \end{cases}$$

You may find this formula useful: if f, g are absolutely continuous functions on $\mathbb{R}_+$ such that $xf(x) \to 0$ and $xg(x) \to 0$ as $x \to \infty$, then

$$\iint_{\mathbb{R}_+^2} f'(x)g'(y)(x \wedge y)\,dx\,dy = \int_0^{\infty} f(x)g(x)\,dx.$$

EXERCISE 2.7. Check Corollary 2.2.

EXERCISE 2.8. Prove Lemma 2.4 by directly calculating $E|W_n - W_n^*|^2$ in terms of a random walk moment.

EXERCISE 2.9. Check the covariance formula (2.27).

References

The results for i.i.d. walks appeared, with a slightly different definition of the current process, in [**Sep05**] and [**Kum08**]. Earlier related results appeared in [**DGL85**].

CHAPTER 3

Independent particles in a random environment

In this chapter we generalize the results of Chapter 2 to particles in a random environment, with the purpose of seeing how the environment influences the outcome. In a fixed environment, that is, conditional on the environment, the particles evolve independently. But under the joint distribution of the walks and the environment, the particles are no longer independent because their evolution gives information about the environment. The environment is static, which means that it is fixed in time.

3.1. Model and results

We formulate the standard one-dimensional nearest-neighbor random walk in random environment (RWRE) model and then put many particles in a fixed environment. We describe the (known) law of large numbers and central limit theorem of the walk itself, and then the (newer) results on current fluctuations for many particles. In this chapter we omit all proofs.

The space of environments is $\Omega = [0,1]^{\mathbb{Z}}$. For an environment $\omega = \{\omega_x\}_{x\in\mathbb{Z}} \in \Omega$ let $\{X^{m,i}_\bullet\}_{m,i}$ be a family of Markov chains on $\mathbb{Z}$ with distribution P_ω determined by the following properties:

(1) $\{X^{m,i}_\bullet\}_{m\in\mathbb{Z},i\in\mathbb{N}}$ are independent under the measure P_ω.
(2) $P_\omega(X^{m,i}_0 = m) = 1$, for all $m \in \mathbb{Z}$ and $i \in \mathbb{N}$.
(3) Each walk obeys these transition probabilities:

$$P_\omega(X^{m,i}_{n+1} = x+1 | X^{m,i}_n = x) = 1 - P_\omega(X^{m,i}_{n+1} = x-1 | X^{m,i}_n = x) = \omega_x.$$

A system of random walks in a random environment may then be constructed by first choosing an environment ω according to a probability distribution P on Ω and then constructing the system of random walks $\{X^{m,i}_\bullet\}$ as described above. The distribution P_ω of the random walks given the environment ω is called the *quenched distribution*. The *averaged distribution* $\mathbb{P}$ (also called *annealed*) is obtained by averaging the quenched law over all environments: $\mathbb{P}(\cdot) = \int_\Omega P_\omega(\cdot) P(d\omega)$. Expectations with respect to the measures P, P_ω and $\mathbb{P}$ are denoted by E_P, E_ω, and $\mathbb{E}$, respectively, and variances with respect to the measure P_ω will be denoted by Var_ω. We make the following assumptions on the environment.

Assumption 1. The distribution P on environments is i.i.d. and uniformly elliptic. That is, $\{\omega_x\}_{x\in\mathbb{Z}}$ are i.i.d. under the measure P, and there exists a $\kappa > 0$ such that $P(\omega_x \in [\kappa, 1-\kappa]) = 1$. Furthermore, $E_P(\rho_0^{2+\varepsilon_0}) < 1$ for some $\varepsilon_0 > 0$, where $\rho_x = (1-\omega_x)/\omega_x$.

These assumptions put the RWRE in the regime where it has transience to $+\infty$ with a strictly positive speed and also satisfies a CLT with an environment-dependent centering. We summarize these results here. Define a shift map on environments by $(\theta^x\omega)_y = \omega_{x+y}$. Let $T_1 = \inf\{n \ge 0 : X_n = 1\}$ be the first hitting time of site $1 \in \mathbb{Z}$ by a RWRE started at the origin, and define

$$Z_{nt}(\omega) = \mathrm{v}_P \sum_{i=0}^{\lfloor nt\mathrm{v}_P\rfloor - 1} (E_{\theta^i\omega}(T_1) - \mathbb{E}T_1).$$

The asymptotic speed v_P is defined in the first statement of the next theorem where we summarize some known basic facts about RWRE.

THEOREM 3.1 ([**Sol75, Pet08, Zei04**]). *Under the assumptions made above we have these conclusions.*

(1) *The RWRE satisfies a law of large numbers with positive speed. That is,*

$$\lim_{n\to\infty}\frac{X_n}{n}=\frac{1-E_P(\rho_0)}{1+E_P(\rho_0)}\equiv \mathrm{v}_P>0, \qquad \mathbb{P}\text{-}a.s. \tag{3.1}$$

(2) *The RWRE satisfies a quenched functional central limit theorem with an environment-dependent centering. Let*

$$B^n(t)=\frac{X_{nt}-nt\mathrm{v}_P+Z_{nt}(\omega)}{\sigma_1\sqrt{n}}, \qquad \text{where } \sigma_1^2=\mathrm{v}_P^3 E_P(\mathrm{Var}_\omega T_1).$$

Then, for P-a.e. environment ω, under the quenched measure P_ω, $B^n(\cdot)$ converges weakly to standard Brownian motion as $n\to\infty$.

(3) *Let*

$$\zeta^n(t)=\frac{Z_{nt}(\omega)}{\sigma_2\sqrt{n}}, \qquad \text{where } \sigma_2^2=\mathrm{v}_P^2\,\mathrm{Var}(E_\omega T_1).$$

Then, under the measure P on environments, $\zeta^n(\cdot)$ converges weakly to standard Brownian motion as $n\to\infty$.

(4) *The RWRE satisfies an averaged central limit theorem. Let*

$$\mathbb{B}^n(t)=\frac{X_{nt}-nt\mathrm{v}_P}{\sigma\sqrt{n}}, \qquad \text{where } \sigma^2=\sigma_1^2+\sigma_2^2.$$

Then, under the averaged measure $\mathbb{P}$, $\mathbb{B}^n(\cdot)$ converges weakly to standard Brownian motion.

The requirement that $E_P(\rho_0^2)<1$ cannot be relaxed in order for the CLT to hold [**KKS75, PZ09**]. Centering by $nt\mathrm{v}_P-Z_{nt}(\omega)$ in the quenched CLT is the same as centering by the quenched mean on account of this bound:

$$\lim_{n\to\infty}P\Big\{\omega:\sup_{k\le n}|E_\omega(X_k)-k\mathrm{v}_P+Z_k(\omega)|\ge\varepsilon\sqrt{n}\Big\}=0, \qquad \forall\varepsilon>0. \tag{3.2}$$

But $Z_{nt}(\omega)$ is more convenient because it is a sum of stationary, ergodic random variables.

These properties of the walk are sufficient for describing the current fluctuations. Assumptions on the initial occupation variables $\eta(0)=\{\eta_x(0)\}$ are similar to those in the previous section. We will allow the distribution of $\eta(0)$ to depend on the environment (in a measurable way), and we assume a certain stationarity.

Assumption 2. Given the environment ω, variables $\{\eta_x(0)\}$ are independent and independent of the random walks. The conditional distribution of $\eta_x(0)$ given ω is denoted by $P_\omega(\eta_x(0)=k)$, and these measurable functions of ω satisfy $P_\omega(\eta_x(0)=k)=P_{\theta^x\omega}(\eta_0(0)=k)$. Also, for some $\varepsilon_0>0$,

$$E_P[\,E_\omega(\eta_x(0))^{2+\varepsilon_0}+\mathrm{Var}_\omega(\eta_x(0))^{2+\varepsilon_0}]<\infty. \tag{3.3}$$

Let

$$\mu_0=E_P[E_\omega(\eta_x(0))]=\mathbb{E}[\eta_x(0)] \quad\text{and}\quad \sigma_0^2=E_P\left[\mathrm{Var}_\omega(\eta_x(0))\right].$$

The current is defined as before:

$$Y_n(t,r) = \sum_{m\le 0}\sum_{k=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,k} > \lfloor nt\mathrm{v}_P\rfloor + r\sqrt{n}\,\}$$
$$- \sum_{m>0}\sum_{k=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,k} \le \lfloor nt\mathrm{v}_P\rfloor + r\sqrt{n}\,\}. \tag{3.4}$$

Now for the results, beginning with the quenched mean of the current. This turns out to essentially follow the correction $Z_{nt}(\omega)$ of the quenched CLT, which is of order $\sqrt{n}$.

Theorem 3.2. *For any $\varepsilon > 0$, $0 < R, T < \infty$,*

$$\lim_{n\to\infty} P\Big\{\omega : \sup_{t\in[0,T],r\in[-R,R]} \big|E_\omega Y_n(t,r) + \mu_0 r\sqrt{n} + \mu_0 Z_{nt}(\omega)\big| \ge \varepsilon\sqrt{n}\,\Big\} = 0. \tag{3.5}$$

Consequently the two-parameter process $\{n^{-1/2}E_\omega Y_n(t,r) : t \in \mathbb{R}_+,\ r \in \mathbb{R}\}$ converges weakly to $\{-\mu_0 r + \mu_0\sigma_2 W(t) : t \in \mathbb{R}_+,\ r \in \mathbb{R}\}$ where $W(\cdot)$ is a standard Brownian motion.

Next we center the current at its quenched mean by defining

$$V_n(t,r) = Y_n(t,r) - E_\omega[Y_n(t,r)].$$

The fluctuations of $V_n(t,r)$ are of order $n^{1/4}$ and similar to the current fluctuations of classical walks from the previous chapter. Recall the definitions of Γ_1 and Γ_2 from (2.9)–(2.10) and abbreviate

$$\Gamma\big((s,q),(t,r)\big) = \mu_0\Gamma_1\big((s,q),(t,r)\big) + \sigma_0^2\Gamma_2\big((s,q),(t,r)\big). \tag{3.6}$$

Let $(V,Z) = (V(t,r), Z(t) : t \in \mathbb{R}_+, r \in \mathbb{R})$ be the process whose joint distribution is defined as follows:

(i) Marginally, $Z(\cdot) = \sigma_2 W(\cdot)$ for a standard Brownian motion $W(\cdot)$.
(ii) Conditionally on the path $Z(\cdot) \in C(\mathbb{R}_+,\mathbb{R})$, V is the mean zero Gaussian process indexed by $\mathbb{R}_+ \times \mathbb{R}$ with covariance

$$\mathbf{E}[V(s,q)V(t,r)\,|\,Z(\cdot)] = \Gamma\big((s,q+Z(s)),(t,r+Z(t))\big) \quad \text{for } (s,q),(t,r) \in \mathbb{R}_+\times\mathbb{R}. \tag{3.7}$$

An equivalent way to say this is to first take independent (V^0, Z) with Z as above and $V^0 = \{V^0(t,r) : (t,r) \in \mathbb{R}_+ \times \mathbb{R}\}$ the mean zero Gaussian process with covariance $\Gamma\big((s,q),(t,r)\big)$ from (3.6), and then define $V(t,r) = V^0(t, r + Z(t))$.

Theorem 3.3. *Under the averaged probability $\mathbb{P}$, as $n \to \infty$, the finite-dimensional distributions of the joint process $\big\{(n^{-1/4}V_n(t,r), n^{-1/2}Z_{nt}(\omega)) : t \in \mathbb{R}_+, r \in \mathbb{R}\big\}$ converge to those of the process (V,Z).*

Thus up to a random shift of the spatial argument we see the same limit process as for classical walks: the process $\bar V(t,r) = V(t, r - Z(t))$ is a mean zero Gaussian process with covariance $\mathbf{E}[\bar V(s,q)\bar V(t,r)] = \Gamma\big((s,q),(t,r)\big)$ from (3.6).

As for classical walks, let us look at the stationary case. The invariant distribution is now valid under a fixed ω: the $\{\eta_x(0)\}$ are independent and

$$\eta_x(0) \sim \text{Poisson}(\mu_0 f(\theta^x\omega)), \quad \text{where} \quad f(\omega) = \frac{\mathrm{v}_P}{\omega_0}\Big(1 + \sum_{i=1}^{\infty}\prod_{j=1}^{i}\rho_j\Big). \tag{3.8}$$

In this case, $E_\omega\eta_0(0) = \mathrm{Var}_\omega\,\eta_0(0) = \mu_0 f(\omega)$. By Assumption 1 $E_P\rho_0^{2+\varepsilon} < 1$ for some $\varepsilon > 0$, and from that it can be shown that $E_P f(\omega)^{2+\varepsilon} < \infty$. Therefore, Assumption 2 holds.

Recall from Corollary 2.2 that for classical random walks the limit process (with fixed space variable r) in the case $\mu_0 = \sigma_0^2$ is fractional Brownian motion ξ with covariance

$$\mathbf{E}[\xi(s)\xi(t)] = \frac{\mu_0\sigma_1}{\sqrt{2\pi}}(\sqrt{s}+\sqrt{t}-\sqrt{|s-t|}\,).$$

For RWRE, $\mu_0 = \sigma_0^2$ implies that

$$\mathbf{E}[V(s,0)V(t,0)\,|\,Z(\cdot)] = \mu_0\big[\Psi_{\sigma_1^2 s}(-Z(s)) + \Psi_{\sigma_1^2 t}(Z(t)) - \Psi_{\sigma_1^2|s-t|}(Z(t)-Z(s))\big]. \tag{3.9}$$

Since the right hand side of (3.9) is a non-constant random variable, the marginal distribution of $V(t,0)$ is non-Gaussian. Taking expectations of (3.9) with respect to $Z(\cdot)$ gives that

$$\mathbf{E}[V(s,0)V(t,0)] = \frac{\mu_0\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}}(\sqrt{s}+\sqrt{t}-\sqrt{|s-t|}). \tag{3.10}$$

Thus we get this conclusion: if $\mu_0 = \sigma_0^2$ for RWRE then the limit process $V(\cdot,0)$ has the same covariance as fractional Brownian motion, but it is not a Gaussian process.

As the reader may have surmised, we can remove the random shift Z from the limit process V by introducing the environment-dependent shift in the current process itself. We state this result too. For $(t,r)\in\mathbb{R}_+\times\mathbb{R}$ define

$$\begin{aligned} Y_n^{(q)}(t,r) = &\sum_{m>0}\sum_{k=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,k} \le nt\mathrm{v}_P - Z_{nt}(\omega) + r\sqrt{n}\,\} \\ &- \sum_{m\le 0}\sum_{k=1}^{\eta_m(0)} \mathbf{1}\{X_{nt}^{m,k} > nt\mathrm{v}_P - Z_{nt}(\omega) + r\sqrt{n}\,\} \end{aligned} \tag{3.11}$$

and its centered version

$$V_n^{(q)}(t,r) = Y_n^{(q)}(t,r) - E_\omega Y_n^{(q)}(t,r).$$

The process $V_n^{(q)}$ has the same limit as classical random walks. Let $V^0 = \{V^0(t,r) : (t,r)\in\mathbb{R}_+\times\mathbb{R}\}$ be the mean zero Gaussian process with covariance $\Gamma\big((s,q),(t,r)\big)$ from (3.6).

THEOREM 3.4. *Under the averaged probability $\mathbb{P}$, as $n\to\infty$, the finite-dimensional distributions of the joint process $\big\{(n^{-1/4}V_n^{(q)}(t,r), n^{-1/2}Z_{nt}(\omega)) : t\in\mathbb{R}_+, r\in\mathbb{R}\big\}$ converge to those of the process (V^0, Z) where V^0 and Z are independent.*

Problems

EXERCISE 3.1. Show the invariance claimed below (3.8). Show also that f is the density of a probability measure that is invariant for the environment process seen by a tagged particle: this is the process $\overline{\omega}_n = \theta^{X_n}\omega$ where the walk starts at $X_0 = 0$.

EXERCISE 3.2. Show that in the stationary case the flux is given by $H(\mu_0) = \mu_0\mathrm{v}_P$.

EXERCISE 3.3. Prove that the random variable $V(t,0)$ discussed around (3.10) in the case $\mu_0 = \sigma_0^2$ is not Gaussian.

References

The results for the current of RWRE's is from [**PS09**]. Zeitouni's lecture notes [**Zei04**] are a standard reference for background on RWRE.

CHAPTER 4

Random average process

4.1. Model and results

The state of the random average process (RAP) is a height function $\sigma : \mathbb{Z} \to \mathbb{R}$ where the value $\sigma(i)$ can be thought of as the height of an interface above site i. The state evolves in discrete time according to the following rule. At each time point $s = 1, 2, 3, \dots$ and at each site $k \in \mathbb{Z}$, a random probability vector $\omega_{k,s} = (\omega_{k,s}(j) : -R \le j \le R)$ of length $2R+1$ is drawn. Given the state $\sigma_{s-1} = (\sigma_{s-1}(i) : i \in \mathbb{Z})$ at time $s-1$, at time s the height value at site k is updated to

$$\sigma_s(k) = \sum_{j:|j|\le R} \omega_{k,s}(j)\sigma_{s-1}(k+j). \tag{4.1}$$

This update is performed independently at each site k to form the state $\sigma_s = (\sigma_s(k) : k \in \mathbb{Z})$ at time s. The weight vectors $\{\omega_{k,s}\}_{k\in\mathbb{Z}, s\in\mathbb{N}}$ are i.i.d. across space-time points (k, s). This system was originally studied by Ferrari and Fontes [**FF98**].

Let

$$p(0, j) = \mathbb{E}[\omega_{0,0}(j)]$$

denote the averaged weights with mean and variance

$$V = \sum_x x\, p(x) \quad \text{and} \quad \sigma_1^2 = \sum_{x\in\mathbb{Z}} (x-V)^2\, p(0,x). \tag{4.2}$$

Let $b = -V$.

Make two nondegeneracy assumptions on the distribution of the weight vectors.

(i) There is no integer $h > 1$ such that, for some $x \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} p(0, x+kh) = 1.$$

This is also expressed by saying that the *span* of the random walk with jump probabilities $p(0, j)$ is 1 [**Dur04**, page 129]. It follows that the additive group generated by $\{x \in \mathbb{Z} : p(0, x) > 0\}$ is all of $\mathbb{Z}$, in other words this walk is *aperiodic* in Spitzer's terminology [**Spi76**].

(ii) Second, we assume that

$$\mathbb{P}\{\max_j \omega_{0,0}(j) < 1\} > 0. \tag{4.3}$$

Let σ_s be a random average process normalized by $\sigma_0(0) = 0$ and whose initial increments $\{\eta_i(0) = \sigma_0(i) - \sigma_0(i-1) : i \in \mathbb{Z}\}$ are i.i.d. such that

$$\text{there exists } \alpha > 0 \text{ such that} \quad E[\,|\eta_i(0)|^{2+\alpha}\,] < \infty. \tag{4.4}$$

As before, the mean and variance of initial increments are

$$\mu_0 = E(\eta_i(0)) \quad \text{and} \quad \sigma_0^2 = \mathrm{Var}(\eta_i(0)).$$

The initial increments η_0 are independent of the weight vectors $\{\omega_{k,s}\}$.

Again we study a suitably scaled process of fluctuations in the characteristic direction: for $(t,r)\in\mathbb{R}_+\times\mathbb{R}$, let

$$\overline{Y}_n(t,r)=n^{-1/4}\big\{\sigma^n_{\lfloor nt\rfloor}(\lfloor r\sqrt{n}\,\rfloor+\lfloor ntb\rfloor)-\mu_0 r\sqrt{n}\,\big\}.$$

In terms of the increment process

$$\eta_i(s)=\sigma_s(i)-\sigma_s(i-1),$$

$\overline{Y}_n(t,r)$ is the centered and scaled net flow from right to left across the path $s\mapsto\lfloor r\sqrt{n}\,\rfloor+\lfloor nsb\rfloor$, during the time interval $0\le s\le t$, exactly as for independent particles.

Recall the definitions (2.9) and (2.10) of the functions Γ_1 and Γ_2.

THEOREM 4.1. *Under the above assumptions the finite-dimensional distributions of the process* $\{\overline{Y}_n(t,r):(t,r)\in\mathbb{R}_+\times\mathbb{R}\}$ *converge weakly as* $n\to\infty$ *to the finite-dimensional distributions of the mean zero Gaussian process* $\{Z(t,r):(t,r)\in\mathbb{R}_+\times\mathbb{R}\}$ *specified by the covariance*

$$\mathbf{E}Z(s,q)Z(t,r)=\mu_0^2\kappa\,\Gamma_1((s,q),(t,r))+\sigma_0^2\Gamma_2((s,q),(t,r)). \tag{4.5}$$

The constant κ is determined by the distribution of the random weights and will be described precisely later in equation (4.29).

Invariant distributions for the general RAP are not known. The next example may be the only one where explicit invariant distributions are available.

EXAMPLE 4.2. Fix positive real parameters $\theta>\alpha>0$. Let $\{\omega_{k,s}(-1):s\in\mathbb{N},k\in\mathbb{Z}\}$ be i.i.d. Beta$(\alpha,\theta-\alpha)$ random variables with density

$$h(u)=\frac{\Gamma(\theta)}{\Gamma(\alpha)\Gamma(\theta-\alpha)}u^{\alpha-1}(1-u)^{\theta-\alpha-1}$$

on $(0,1)$. Set $\omega_{k,s}(0)=1-\omega_{k,s}(-1)$. Thus the weights are supported on $\{-1,0\}$. A family of invariant distributions for the increment process $\eta(s)=(\eta_k(s):k\in\mathbb{Z})$ is obtained by letting the variables $\{\eta_k:k\in\mathbb{Z}\}$ be i.i.d. Gamma(θ,λ) distributed with common density

$$f(x)=\frac{1}{\Gamma(\theta)}\lambda e^{-\lambda x}(\lambda x)^{\theta-1} \tag{4.6}$$

on $\mathbb{R}_+$. This family of invariant distributions is parametrized by $0<\lambda<\infty$. Under this distribution $E^\lambda[\eta_k]=\theta/\lambda$ and $\mathrm{Var}^\lambda[\eta_k]=\theta/\lambda^2$. In this situation we find again the fractional Brownian motion limit:

$$\mathbf{E}Z(s,0)Z(t,0)=c_1\big(\sqrt{s}+\sqrt{t}-\sqrt{|t-s|}\,\big). \tag{4.7}$$

for a certain constant c_1.

4.2. Steps of the proof

1. Representation in terms of space-time RWRE

Let $\omega=(\omega_{k,s}:s\in\mathbb{N},\,k\in\mathbb{Z})$ represent the i.i.d. random weight vectors that determine the dynamics, coming from a probability space $(\Omega,\mathfrak{S},\mathbb{P})$. Given ω and a space-time point (i,τ), let $\{X_s^{i,\tau}:s\in\mathbb{Z}_+\}$ denote a random walk on $\mathbb{Z}$ that starts at $X_0^{i,\tau}=i$ and whose transition probabilities are given by

$$P^\omega(X^{i,\tau}_{s+1}=y\,|\,X^{i,\tau}_s=x)=\omega_{x,\tau-s}(y-x). \tag{4.8}$$

P^ω is the path measure of the walk $X_s^{i,\tau}$, with expectation denoted by E^ω. Comparison of (4.1) and (4.8) gives

$$\sigma_s(i) = \sum_j P^\omega(X_1^{i,\,s} = j \mid X_0^{i,\,s} = i)\sigma_{s-1}(j) = E^\omega\big[\sigma_{s-1}(X_1^{i,\,s})\big]. \tag{4.9}$$

Iteration and the Markov property of the walks $X_s^{i,s}$ then lead to

$$\sigma_s(i) = E^\omega\big[\sigma_0(X_s^{i,\,s})\big]. \tag{4.10}$$

Note that the initial height function σ_0 is a constant under the expectation E^ω.

Let us add another coordinate to keep track of time and write $\overline{X}_s^{i,\tau} = (X_s^{i,\tau}, \tau - s)$ for $s \geq 0$. Then $\overline{X}_s^{i,\tau}$ is a random walk on the planar lattice $\mathbb{Z}^2$ that always moves down one step in the e_2-direction, and if its current position is (x,n), then its next position is $(x+y, n-1)$ with probability $\omega_{x,n}(y-x)$. We could call this a backward random walk in a (space-time, or dynamical) random environment.

The opening step of the proof is to use the random walk representation to rewrite the random variable $\overline{Y}_n(t,r)$ in a manner that allows us to separate the effects of the random initial conditions from the effects of the random weights. Abbreviate

$$y(n) = \lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor.$$

and recall $\mu_0 = E\eta_i(0)$ and $\sigma_0(0) = 0$.

$$\begin{aligned}
\sigma_{\lfloor nt\rfloor}(y(n)) &= E^\omega\big[\sigma_0(X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor})\big] \\
&= E^\omega\Bigg[\mathbf{1}_{\left\{X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}>0\right\}} \sum_{i=1}^{X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}} \eta_i(0) \;-\; \mathbf{1}_{\left\{X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}<0\right\}} \sum_{i=X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}+1}^{0} \eta_i(0)\Bigg] \\
&= \sum_{i>0} \eta_i(0) P^\omega\left\{i \leq X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\right\} \;-\; \sum_{i\leq 0} \eta_i(0) P^\omega\left\{i > X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\right\} \\
&= \mu_0 H_n(t,r) + S_n(t,r)
\end{aligned}$$

where

$$\begin{aligned}
H_n(t,r) &= \sum_{i\in\mathbb{Z}} \Big(\mathbf{1}\{i>0\} P^\omega\big\{i \leq X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\big\} \;-\; \mathbf{1}\{i\leq 0\} P^\omega\big\{i > X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\big\}\Big) \\
&= E^\omega\big(X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\big)
\end{aligned}$$

and

$$S_n(t,r) = \sum_{i\in\mathbb{Z}} \big(\eta_i(0) - \mu_0\big)\Big(\mathbf{1}\{i>0\} P^\omega\big\{i \leq X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\big\} \;-\; \mathbf{1}\{i\leq 0\} P^\omega\big\{i > X_{\lfloor nt\rfloor}^{y(n),\,\lfloor nt\rfloor}\big\}\Big).$$

At this point the terms H_n and S_n are dependent, but in the course of the scaling limit they become independent and furnish the two independent pieces that make up the limiting process Z. The limits $n^{-1/4}(H_n(t,r) - r\sqrt{n}) \xrightarrow{\mathcal{D}} H(t,r)$ and $n^{-1/4}S_n(t,r) \xrightarrow{\mathcal{D}} S(t,r)$ are treated separately, and then together

$$\overline{Y}_n = n^{-1/4}(H_n - r\sqrt{n} + S_n) \xrightarrow{\mathcal{D}} H + S \equiv Z$$

with independent terms H and S. This independence comes from the independence of the initial height function σ_0 and the random environment ω that drives the dynamics. The idea is represented in the next lemma.

Lemma 4.3. *Let η and ω be independent random variables with values in some abstract measurable spaces. Let $h_n(\omega)$ and $s_n(\omega,\eta)$ be measurable functions of (ω,η). Let $E^\omega(\cdot)$ denote conditional expectation, given ω. Assume the existence of random variables h and s such that*

(i) $h_n(\omega) \xrightarrow{\mathcal{D}} h$;
(ii) *for all* $\theta \in \mathbb{R}$, $E^\omega[e^{i\theta s_n}] \to E(e^{i\theta s})$ *in probability as* $n \to \infty$.

Then $h_n + s_n \xrightarrow{\mathcal{D}} h + s$*, where h and s are independent.*

Proof. Let $\theta, \lambda \in \mathbb{R}$. Then

$$\begin{aligned}&\left|E\big(E^\omega[e^{i\lambda h_n + i\theta s_n}]\big) - E[e^{i\lambda h}]\,E[e^{i\theta s}]\right|\\ &\quad\le \left|E\left[e^{i\lambda h_n}\left(E^\omega e^{i\theta s_n} - Ee^{i\theta s}\right)\right]\right| + \left|\left(Ee^{i\lambda h_n} - Ee^{i\lambda h}\right)Ee^{i\theta s}\right|\\ &\quad\le \left|E\left[e^{i\lambda h_n}\left(E^\omega e^{i\theta s_n} - Ee^{i\theta s}\right)\right]\right| + \left|Ee^{i\lambda h_n} - Ee^{i\lambda h}\right|.\end{aligned}$$

By assumption (i), the second term above goes to 0. By assumption (ii), the integrand in the first term goes to 0 in probability. Therefore by bounded convergence the first term goes to 0 as $n \to \infty$. ☐

We discuss the term S_n briefly and reserve most of our attention to H_n. Two limits combine to give the result. The idea is to apply the Lindeberg-Feller theorem to $S_n(t,r)$ under a fixed ω. Then the ω-dependent coefficients provide no fluctuations but instead converge to Brownian probabilities due to a quenched central limit theorem for the space-time RWRE. Here is an informal presentation where we first imagine that the coefficients can be replaced by deterministic quantities:

$$\begin{aligned}S_n(t,r) &= \sum_{x\in\mathbb{Z}}\big(\eta_x(0)-\mu_0\big)\Bigg(\mathbf{1}\{x>0\}P^\omega\Bigg\{\frac{X_{\lfloor nt\rfloor}^{\lfloor ntb\rfloor+\lfloor r\sqrt{n}\rfloor,\,\lfloor nt\rfloor} - r\sqrt{n}}{\sqrt{n}} \ge \frac{x}{\sqrt{n}} - r\Bigg\}\\ &\quad-\ \mathbf{1}\{x\le 0\}P^\omega\Bigg\{\frac{X_{\lfloor nt\rfloor}^{\lfloor ntb\rfloor+\lfloor r\sqrt{n}\rfloor,\,\lfloor nt\rfloor} - r\sqrt{n}}{\sqrt{n}} < \frac{x}{\sqrt{n}} - r\Bigg\}\Bigg)\\ &\approx \sum_{x\in\mathbb{Z}}\big(\eta_x(0)-\mu_0\big)\Bigg(\mathbf{1}\{x>0\}\mathbf{P}\Big\{B_{\sigma_1^2 t} > \frac{x}{\sqrt{n}} - r\Big\}\ -\ \mathbf{1}\{x\le 0\}\mathbf{P}\Big\{B_{\sigma_1^2 t} \le \frac{x}{\sqrt{n}} - r\Big\}\Bigg)\end{aligned}$$

Now apply the Lindeberg-Feller theorem to the remaining sum of independent initial occupation variables, and the limiting covariance comes as:

$$\begin{aligned}&n^{-1/2}\sum_{i,j}\theta_i\theta_j E[S_n(t_i,r_i)S_n(t_j,r_j)]\\ &\approx \sigma_0^2\sum_{i,j}\theta_i\theta_j\, n^{-1/2}\Bigg[\sum_{x>0}\mathbf{P}\Big\{B_{\sigma_1^2 t_i} > \frac{x}{\sqrt{n}} - r_i\Big\}\mathbf{P}\Big\{B_{\sigma_1^2 t_j} > \frac{x}{\sqrt{n}} - r_j\Big\}\\ &\qquad\qquad + \sum_{x\le 0}\mathbf{P}\Big\{B_{\sigma_1^2 t_i} \le \frac{x}{\sqrt{n}} - r_i\Big\}\mathbf{P}\Big\{B_{\sigma_1^2 t_j} \le \frac{x}{\sqrt{n}} - r_j\Big\}\Bigg]\\ &\approx \sigma_0^2\sum_{i,j}\theta_i\theta_j\Bigg[\int_0^\infty \mathbf{P}\{B_{\sigma_1^2 t_i} > x - r_i\}\mathbf{P}\{B_{\sigma_1^2 t_j} > x - r_j\}\,dx\\ &\qquad\qquad + \int_{-\infty}^0 \mathbf{P}\{B_{\sigma_1^2 t_i} \le x - r_i\}\mathbf{P}\{B_{\sigma_1^2 t_j} \le x - r_j\}\,dx\Bigg]\end{aligned}$$

$$= \sigma_0^2 \sum_{i,j} \theta_i \theta_j \Gamma_2((t_i, r_i), (t_j, r_j)).$$

Turning this argument rigorous gives the limit in $\mathbb{P}$-probability:

$$\lim_{n\to\infty} E^\omega\big[e^{i\sum_k \theta_k S_n(t_k,r_k)}\big] = E\big[e^{i\sum_k \theta_k S(t_k,r_k)}\big] = \exp\Big\{-\tfrac{1}{2}\sigma_0^2 \sum_{i,j} \theta_i\theta_j \Gamma_2((t_i,r_i),(t_j,r_j))\Big\}. \tag{4.11}$$

In particular, in the limit the fluctuations of S come from the initial occupation variables $\eta_i(0)$ and hence are independent of the weights ω that determine H_n.

2. Quenched mean of the backward space-time RWRE

The remaining piece of the fluctuations comes from

$$\bar{H}_n(t,r) = n^{-1/4} E^\omega\big(X_{\lfloor nt\rfloor}^{\lfloor ntb\rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt\rfloor} - \lfloor r\sqrt{n}\,\rfloor\big) \tag{4.12}$$

THEOREM 4.4. *In the sense of convergence of finite-dimensional distributions, $\bar{H}_n \xrightarrow{\mathcal{D}} H$ where $H(t,r)$ is the mean zero Gaussian process with covariance*

$$\mathbf{E}H(s,q)H(t,r) = \kappa\Gamma_1((s,q),(t,r)) = \frac{\kappa}{2}\int_{\sigma_1^2|t-s|}^{\sigma_1^2(t+s)} \frac{1}{\sqrt{2\pi v}} \exp\Big\{-\frac{1}{2v}(q-r)^2\Big\}\, dv. \tag{4.13}$$

The constant κ is defined below in (4.29).

By comparing covariances (Exercise 2.5) one checks that H can also be defined by

$$H(t,r) = \sqrt{\kappa} \iint_{[0,t]\times\mathbb{R}} \varphi_{\sigma_1^2(t-s)}(r-z)\, dW(s,z). \tag{4.14}$$

Formula (4.14) implies that process $\{H(t,r)\}$ is a weak solution for this initial value problem of the stochastic heat equation:

$$H_t = \tfrac{\sigma_1^2}{2} H_{rr} + \sqrt{\kappa}\,\dot{W}\,, \qquad H(0,r) \equiv 0. \tag{4.15}$$

Proof of Theorem 4.4 happens in two steps: first for multiple space points at a fixed time, and then across time points.

Step 1. Martingale increments for fixed time.

Let us abbreviate $X_s^* = X_s^{\lfloor ntb\rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt\rfloor}$. Note that

$$E(X^*_{\lfloor nt\rfloor}) = \lfloor ntb\rfloor + \lfloor r\sqrt{n}\,\rfloor + \lfloor nt\rfloor V = r\sqrt{n} + O(1)$$

so $\bar{H}(t,r)$ in (4.12) is essentially centered and we can pretend that it is exactly centered. Let

$$g(\omega) = E^\omega(X_1^{0,0}) - V$$

be the centered local drift. Recall the space-time walk $\overline{X}_m^{x,s} = (X_m^{x,s}, s-m)$. By the Markov property of the walk

$$\begin{aligned} E^\omega(X_n^{x,s}) - x - nV &= \sum_{k=0}^{n-1} E^\omega\big[X_{k+1}^{x,s} - X_k^{x,s} - V\big] \\ &= \sum_{k=0}^{n-1} E^\omega\big[E^{T_{\{\overline{X}_k^{x,s}\}}\omega}(X_1^{0,0} - V)\big] = \sum_{k=0}^{n-1} E^\omega g(T_{\overline{X}_k^{x,s}}\omega). \end{aligned} \tag{4.16}$$

$(T_{x,m}\omega)_{y,s} = \omega_{x+y,m+s}$ is the space-time shift of environments. The g-terms above are martingale increments under the distribution $\mathbb{P}$ of the environments, relative to the filtration defined by levels of environments: writing $\bar\omega_{m,n} = \{\omega_{x,s} : x \in \mathbb{Z}, m \le s \le n\}$, and with fixed (x,m) and time $n = 0, 1, 2\ldots,$

$$\mathbb{E}\big[E^\omega g(T_{\overline{X}_n^{x,m}}\omega)\big|\bar\omega_{m-n+1,m}\big] = \sum_{y\in\mathbb{Z}} P^\omega\{\overline{X}_n^{x,m} = (y, m-n)\} \int g(T_{y,m-n}\omega)\,\mathbb{P}(d\bar\omega_{m-n}) = 0.$$

The point above is that the probability $P^\omega\{\overline{X}_n^{x,m} = (y, m-n)\}$ is determined by $\bar\omega_{m-n+1,m}$.

It turns out that we can apply a martingale central limit theorem to conclude that, for a fixed t, a vector

$$\big(\bar H_n(t, r_1), \bar H_n(t, r_2), \ldots, \bar H_n(t, r_N)\big)$$

becomes a Gaussian vector in the $n \to \infty$ limit. Let us take this for granted, and compute the covariance of the limit. This leads us to study an auxiliary Markov chain which has been useful for space-time (and more general ballistic) RWRE.

Given points (t,q) and (t,r), abbreviate $X_s^{(1)} = X_s^{\lfloor ntb\rfloor+\lfloor q\sqrt{n}\rfloor,\,\lfloor nt\rfloor}$ and $X_s^{(2)} = X_s^{\lfloor ntb\rfloor+\lfloor r\sqrt{n}\rfloor,\,\lfloor nt\rfloor}$.

$$\begin{aligned}
\mathbb{E}\big[\bar H_n(t,q)\bar H_n(t,r)\big] &= n^{-1/2}\mathbb{E}\Big[E^\omega\big(X^{(1)}_{\lfloor nt\rfloor} - \lfloor q\sqrt{n}\rfloor\big)E^\omega\big(X^{(2)}_{\lfloor nt\rfloor} - \lfloor r\sqrt{n}\rfloor\big)\Big]\\
&= n^{-1/2}\mathbb{E}\Big[\Big(\sum_{j=0}^{\lfloor nt\rfloor-1} E^\omega g(T_{\overline{X}^{(1)}_j}\omega)\Big)\Big(\sum_{k=0}^{\lfloor nt\rfloor-1} E^\omega g(T_{\overline{X}^{(2)}_k}\omega)\Big)\Big]\\
&= n^{-1/2}\sum_{j,k}\sum_{x,y}\mathbb{E}\Big[P^\omega(X^{(1)}_j = x)P^\omega(X^{(2)}_k = y)g(T_{x,\lfloor nt\rfloor-j}\omega)g(T_{y,\lfloor nt\rfloor-k}\omega)\Big]\\
&= \sigma_D^2 n^{-1/2}\sum_{k=0}^{\lfloor nt\rfloor-1}\mathbb{E}P^\omega(X^{(1)}_k = X^{(2)}_k).
\end{aligned}$$

The last step uses the independence of the environment. We denote the variance of the drift by $\sigma_D^2 = \mathbb{E}(g^2)$. Let now $Y_k = X_k^{(2)} - X_k^{(1)}$ be the difference of two independent walks in a common environment. Y_k is a Markov chain on $\mathbb{Z}$ with transition probability

$$q(x,y) = \begin{cases}\displaystyle\sum_{z\in\mathbb{Z}}\mathbb{E}[\omega_{0,0}(z)\omega_{0,0}(0, z+y)] & x = 0\\ \displaystyle\sum_{z\in\mathbb{Z}} p(0,z)p(0,z+y-x) & x \ne 0.\end{cases}$$

Y_n can be thought of as a symmetric random walk on $\mathbb{Z}$ whose transition has been perturbed at the origin. The corresponding homogeneous, unperturbed transition probabilities are

$$\bar q(x,y) = \bar q(0, y-x) = \sum_{z\in\mathbb{Z}} p(0,z)p(0,z+y-x) \qquad (x, y\in\mathbb{Z}).$$

Continuing from above, with $x_n = \lfloor r\sqrt{n}\rfloor - \lfloor q\sqrt{n}\rfloor$,

$$\mathbb{E}\big[\bar H_n(t,q)\bar H_n(t,r)\big] = \frac{\sigma_D^2}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1} q^k(x_n, 0) = \frac{\sigma_D^2}{\sqrt{n}} G_{\lfloor nt\rfloor-1}(x_n, 0). \tag{4.17}$$

If Y_k were a symmetric random walk, we would know this limit exactly from the local central limit theorem:

LEMMA 4.5. *For a mean* 0*, span* 1 *random walk S_n on $\mathbb{Z}$ with finite variance σ^2, $a \in \mathbb{R}$ and points $a_n \in \mathbb{Z}$ such that $|a_n - a\sqrt{n}| = O(1)$,*

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1} P(S_k = a_n) = \frac{1}{\sigma^2}\int_0^{\sigma^2 t}\frac{1}{\sqrt{2\pi v}}\exp\Big\{-\frac{a^2}{2v}\Big\}\,dv.$$

Proof. By the local CLT [**Dur04**, Section 2.5]

$$\lim_{m\to\infty}\sup_{x\in\mathbb{Z}}\sqrt{m}\,\Big|P(S_m=x)-\frac{1}{\sqrt{2\pi m\sigma^2}}\exp\Big\{-\frac{x^2}{2m\sigma^2}\Big\}\Big|=0.$$

Use this in a Riemann sum argument (details as exercise). □

For the homogeneous $\bar{q}$-walk this lemma gives (using symmetry)

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=0}^{\lfloor nt\rfloor-1}\bar{q}^k(x_n,0)=\frac{1}{2\sigma_1^2}\int_0^{2\sigma_1^2 t}\frac{1}{\sqrt{2\pi v}}\exp\Big\{-\frac{(r-q)^2}{2v}\Big\}\,dv. \tag{4.18}$$

Now the task is to relate the transitions q and $\bar{q}$. For this purpose we introduce one more player: the potential kernel of the symmetric $\bar{q}$-walk, defined by

$$\bar{a}(x)=\lim_{n\to\infty}\big[\bar{G}_n(0,0)-\bar{G}_n(x,0)\big]=\lim_{n\to\infty}\Big\{\sum_{k=0}^{n}\bar{q}^k(0,0)-\sum_{k=0}^{n}\bar{q}^k(x,0)\Big\}. \tag{4.19}$$

The potential kernel satisfies $\bar{a}(0)=0$, the equations

$$\bar{a}(x)=\sum_{y\in\mathbb{Z}}\bar{q}(x,y)\bar{a}(y)\quad\text{for } x\neq 0\text{, and}\quad\sum_{y\in\mathbb{Z}}\bar{q}(0,y)\bar{a}(y)=1, \tag{4.20}$$

and the limit

$$\lim_{x\to\pm\infty}\frac{\bar{a}(x)}{|x|}=\frac{1}{2\sigma_1^2}. \tag{4.21}$$

(For existence of $\bar{a}$ and its properties, see [**Spi76**, Sections 28-29].)

Example 4.6. If for some $k\in\mathbb{Z}$, $p(0,k)+p(0,k+1)=1$, so that $\bar{q}(0,x)=0$ for $x\notin\{-1,0,1\}$, then $\bar{a}(x)=|x|/(2\sigma_a^2)$.

Define the constant

$$\beta=\sum_{x\in\mathbb{Z}}q(0,x)\bar{a}(x). \tag{4.22}$$

This constant accounts for the difference in the limits of the Green's functions for transitions q and $\bar{q}$.

Lemma 4.7. *Let $x\in\mathbb{R}$ and $x_n\in\mathbb{Z}$ be such that $x_n-n^{1/2}x$ stays bounded. Then*

$$\lim_{n\to\infty}n^{-1/2}G_n\big(x_n,0\big)=\frac{1}{2\beta\sigma_1^2}\int_0^{2\sigma_1^2}\frac{1}{\sqrt{2\pi v}}\exp\Big\{-\frac{x^2}{2v}\Big\}\,dv. \tag{4.23}$$

Proof. Given limit (4.18), it suffices to prove

$$\sup_{z\in\mathbb{Z}}\Big|\frac{\beta}{\sqrt{n}}G_n(z,0)\;-\;\frac{1}{\sqrt{n}}\bar{G}_n(z,0)\Big|\to 0. \tag{4.24}$$

First we prove the case $z=0$.

By (4.18),

$$\lim_{n\to\infty}n^{-1/2}\bar{G}_n(0,0)=\frac{1}{\sqrt{\pi\sigma_1^2}}\,. \tag{4.25}$$

We need to show

$$\lim_{n\to\infty}n^{-1/2}G_n(0,0)=\frac{1}{\beta\sqrt{\pi\sigma_1^2}}. \tag{4.26}$$

Using (4.20), $\bar a(0)=0$, and $\bar q(x,y)=q(x,y)$ for $x\neq 0$,

$$\begin{aligned}\sum_{x\in\mathbb{Z}} q^m(0,x)\bar a(x) &= \sum_{x\neq 0} q^m(0,x)\bar a(x) = \sum_{x\neq 0, y\in\mathbb{Z}} q^m(0,x)\bar q(x,y)\bar a(y)\\ &= \sum_{x\neq 0, y\in\mathbb{Z}} q^m(0,x)q(x,y)\bar a(y) = \sum_{y\in\mathbb{Z}} q^{m+1}(0,y)\bar a(y) - q^m(0,0)\sum_{y\in\mathbb{Z}} q(0,y)\bar a(y).\end{aligned}$$

Constant β appears in the last term. Sum over $m=0,1,\dots,n-1$ to get

$$\big(1+q(0,0)+\cdots+q^{n-1}(0,0)\big)\beta = \sum_{x\in\mathbb{Z}} q^n(0,x)\bar a(x)$$

and write this in the form

$$n^{-1/2}\beta G_{n-1}(0,0) = n^{-1/2}E_0\big[\bar a(Y_n)\big].$$

Recall that $Y_n = X_n - \tilde X_n$ where X_n and $\tilde X_n$ are two independent walks in the same environment. By the quenched CLT for space-time RWRE, $n^{-1/2}Y_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,2\sigma_1^2)$. Marginally X_n and $\tilde X_n$ are i.i.d. walks with bounded steps, hence there is enough uniform integrability to conclude that

$$n^{-1/2}E_0|Y_n| \to 2\sqrt{\sigma_1^2/\pi}.$$

By (4.21) and some estimation (exercise),

$$n^{-1/2}E_0\big[\bar a(Y_n)\big] \to \frac{1}{\sqrt{\sigma_1^2\pi}}\,. \tag{4.27}$$

This proves (4.26) and thereby limit (4.24) for $z=0$.

To get the full statement in (4.24), for $k\geq 1$ and $z\neq 0$ let

$$f^k(z,0) = \mathbf{1}_{\{z\neq 0\}} \sum_{z_1\neq 0,\dots,z_{k-1}\neq 0} q(z,z_1)q(z_1,z_2)\cdots q(z_{k-1},0)$$

denote the probability that the first visit to the origin occurs at time k. This quantity is the same for both q and $\bar q$ because these processes do not differ until the origin is visited. Choose n_0 so that

$$|\beta G_m(0,0) \,-\, \bar G_m(0,0)| \leq \varepsilon\sqrt{m} \quad \text{for } m\geq n_0.$$

Then

$$\begin{aligned}&\sup_{z\neq 0}\Big|\frac{\beta}{\sqrt n}G_n(z,0) \,-\, \frac{1}{\sqrt n}\bar G_n(z,0)\Big|\\ &\leq \sup_{z\neq 0}\frac{1}{\sqrt n}\sum_{k=1}^{n} f^k(z,0)\big|\beta G_{n-k}(0,0) \,-\, \bar G_{n-k}(0,0)\big|\\ &\leq \sup_{z\neq 0}\frac{\varepsilon}{\sqrt n}\sum_{k=1}^{n-n_0} f^k(z,0)\sqrt{n-k} + \frac{Cn_0^2}{\sqrt n} \leq \varepsilon + \frac{Cn_0^2}{\sqrt n}.\end{aligned}$$

Letting $n\to\infty$ completes the proof. ∎

Combining (4.17) and (4.23) gives

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\big[\bar H_n(t,q)\bar H_n(t,r)\big] &= \frac{\sigma_D^2}{2\beta\sigma_1^2}\int_0^{2\sigma_1^2}\frac{1}{\sqrt{2\pi v}}\exp\Big\{-\frac{x^2}{2v}\Big\}\,dv\\ &= \kappa\Gamma_1((t,q),(t,r)),\end{aligned} \tag{4.28}$$

where we defined a the new constant

$$\kappa = \frac{\sigma_D^2}{\beta \sigma_1^2}. \tag{4.29}$$

While we have not furnished all the details, let us consider proved that for a fixed t, the finite-dimensional distributions of $\bar{H}(t,r)$ converge to the Gaussian process $H(t,r)$ with covariance $\kappa\Gamma_1((t,q),(t,r))$.

Step 2. Markov property for time steps.

This step is overly technical and so we only give a sketch of the idea behind it. Stopping and restarting the walk $X_{\bullet}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor}$ at level $\lfloor ns \rfloor$ gives:

$$\begin{aligned}
&E^\omega\big(X_{\lfloor nt \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor}\big) - \lfloor r\sqrt{n}\,\rfloor \\
&= \sum_{x\in\mathbb{Z}} P^\omega\big(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor} = \lfloor nsb \rfloor + x\big) E^\omega\big(X_{\lfloor ns \rfloor}^{\lfloor nsb \rfloor + x,\, \lfloor ns \rfloor}\big) - \lfloor r\sqrt{n}\,\rfloor \\
&= \sum_{x\in\mathbb{Z}} P^\omega\big(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor} = \lfloor nsb \rfloor + x\big) \big[\, E^\omega\big(X_{\lfloor ns \rfloor}^{\lfloor nsb \rfloor + x,\, \lfloor ns \rfloor}\big) - x\,\big] \\
&\qquad + E^\omega\big(X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor}\big) - \lfloor nsb \rfloor - \lfloor r\sqrt{n}\,\rfloor.
\end{aligned}$$

Change summation index to $u = x/\sqrt{n}$. Then we have approximately the identity

$$\bar{H}_n(t,r) = \sum_{u\in n^{-1/2}\mathbb{Z}} P^\omega\left\{ \frac{X_{\lfloor nt \rfloor - \lfloor ns \rfloor}^{\lfloor ntb \rfloor + \lfloor r\sqrt{n}\,\rfloor,\, \lfloor nt \rfloor} - \lfloor nsb \rfloor - r\sqrt{n}}{\sqrt{n}} = u - r \right\} \bar{H}_n(s,u) + \bar{H}_n^*(t-s,r),$$

where $\bar{H}_n^*(t-s,r)$ is the same as $\bar{H}_n(t-s,r)$ but with origin shifted (approximately) to (nsb, ns). On the right-hand side, the processes $\bar{H}_n(s,\,\cdot)$ and $\bar{H}_n^*(t-s,\,\cdot)$ are independent of each other because they depend on disjoint levels of environments. (This is best seen with the help of a picture.) As $n\to\infty$ the probability coefficients of the sum converge to deterministic Gaussian probabilities by the quenched CLT for the RWRE. By the result for fixed t, the right-hand side above converges in distribution.

Taking the limits and supplying all the technicalities leads to the equation

$$H(t,r) = \int_{\mathbb{R}} \varphi_{\sigma_1^2(t-s)}(u-r) H(s,u)\,du + H^*(t-s,r)$$

where on the right, the processes $H(s,\,\cdot)$ and $H^*(t-s,\,\cdot)$ are independent. From this equation one can verify that the finite-dimensional distributions of the process $H(t,r)$ are Gaussian with covariance $\kappa\Gamma_1((s,q),(t,r))$ as stated in Theorem 4.4.

This concludes the presentation of the random average process limit.

Problems

Exercise 4.1. (a) Suppose the weights are supported on $\{-1,0\}$ as in Example 4.2. Derive the equation for computing the increments $\{\eta_s(k) : k\in\mathbb{Z}\}$ from the increments at the previous time $\{\eta_{s-1}(k) : k\in\mathbb{Z}\}$.

(b) Let $\eta\sim\text{Gamma}(\theta,\lambda)$ and $U\sim\text{Beta}(\alpha,\theta-\alpha)$ be independent. Set $X = U\eta$ and $Y=(1-U)\eta$. Show that X and Y are independent with distributions $X\sim\text{Gamma}(\alpha,\lambda)$, $Y\sim\text{Gamma}(\theta-\alpha,\lambda)$.

(c) Verify the invariance claim made in Example 4.2.

Exercise 4.2. Derive (4.10).

EXERCISE 4.3. Finnish the proof of Lemma 4.5.

EXERCISE 4.4. Derive (4.27).

References

The fluctuation results for RAP presented here are from [**BRAS06**]. In addition to [**FF98**], RAP was later studied also in [**FMV03**]. The quenched CLT for space-time RWRE has been proved several times with progressively better assumptions, see [**RAS05**].

CHAPTER 5

Asymmetric simple exclusion process

The asymmetric simple exclusion process (ASEP) is a Markov process that describes the motion of particles on the one-dimensional integer lattice $\mathbb{Z}$. Each particle executes a continuous-time nearest-neighbor random walk on $\mathbb{Z}$ with jump rate p to the right and q to the left. Particles interact through the exclusion rule which means that at most one particle is allowed at each site. Any attempt to jump onto an already occupied site is prevented from happening. The asymmetric case is $p \neq q$. We assume $0 \leq q < p \leq 1$ and $p + q = 1$.

For this process we do not derive precise distributional limits for the current, but only bounds that reveal the order of magnitude of the fluctuations. In contrast with the earlier results for linear flux, the magnitude of current fluctuations is now $t^{1/3}$.

The proofs of these bounds are based on couplings, and make heavy use of the notion of *second class particle*.

5.1. Basic properties

We run quickly through the fundamentals of (p, q)-ASEP.

Definition and graphical construction. The state of the system at time t is a configuration $\eta(t) = (\eta_i(t))_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ of zeroes and ones. The value $\eta_i(t) = 1$ means that site i is occupied by a particle at time t, while the value $\eta_i(t) = 0$ means that site i is vacant at time t.

The motion of the particles is controlled by independent Poisson processes (*Poisson clocks*) $\{N^{i\to i+1}, N^{i\to i-1} : i \in \mathbb{Z}\}$ on $\mathbb{R}_+$. These Poisson processes are independent of the (possibly random) initial configuration $\eta(0)$. Each Poisson clock $N^{i\to i+1}$ has rate p and each $N^{i\to i-1}$ has rate q. If t is a jump time for $N^{i\to i+1}$ and if $(\eta_i(t-), \eta_{i+1}(t-)) = (1, 0)$ then at time t the particle from site i moves to site $i + 1$ and the new values are $(\eta_i(t), \eta_{i+1}(t)) = (0, 1)$. Similarly if t is a jump time for $N^{i\to i-1}$ a particle is moved from i to $i - 1$ at time t, provided the configuration at time $t-$ permits this move. If the jump prompted by a Poisson clock is not permitted by the state of the system, this jump attempt is simply ignored and the particles resume waiting for the next prompt coming from the Poisson clocks.

This construction of the process is known as the *graphical construction* or the *Harris construction*. When the initial state is a fixed configuration η, P^η denotes the distribution of the process.

We write η, ω, etc for elements of the state space $\{0, 1\}^{\mathbb{Z}}$, but also for the entire process so that η-process stands for $\{\eta_i(t) : i \in \mathbb{Z}, 0 \leq t < \infty\}$. The configuration δ_i is the state that has a single particle at position i but otherwise the lattice is vacant.

Invariant distributions. A basic fact is that i.i.d. Bernoulli distributions $\{\nu^\rho\}_{\rho\in[0,1]}$ are extremal invariant distributions for ASEP. For each density value $\rho \in [0, 1]$, ν^ρ is the probability measure on $\{0, 1\}^{\mathbb{Z}}$ under which the occupation variables $\{\eta_i\}$ are i.i.d. with common mean $\int \eta_i \, d\nu^\rho = \rho$. When the process η is stationary with time-marginal ν^ρ, we write P^ρ for the probability distribution of the entire process. The *stationary density-ρ process* means the ASEP η that is stationary in time and has marginal distribution $\eta(t) \sim \nu_\rho$ for all $t \in \mathbb{R}_+$.

Basic coupling and second class particles. The *basic coupling* of two exclusion processes η and ω means that they obey a common set of Poisson clocks $\{N^{i\to i+1}, N^{i\to i-1}\}$. Suppose the two processes η and η^+ satisfy $\eta^+(0) = \eta(0) + \delta_{Q(0)}$ at time zero, for some position $Q(0) \in \mathbb{Z}$. This means that $\eta_i^+(0) = \eta_i(0)$ for all $i \neq Q(0)$, $\eta^+_{Q(0)}(0) = 1$ and $\eta_{Q(0)}(0) = 0$. Then throughout the evolution in the basic coupling there is a single discrepancy between $\eta(t)$ and $\eta^+(t)$ at some position $Q(t)$: $\eta^+(t) = \eta(t) + \delta_{Q(t)}$. From the perspective of $\eta(t)$, $Q(t)$ is called a second class particle. By the same token, from the perspective of $\eta^+(t)$, $Q(t)$ is a second class *anti*particle. In particular, we shall call the pair (η, Q) *a (p,q)-ASEP with a second class particle*.

We write a boldface $\mathbf{P}$ for the probability measure when more than one process are coupled together. In particular, $\mathbf{P}^\rho$ represents the situation where the initial occupation variables $\eta_i(0) = \eta_i^+(0)$ are i.i.d. mean-ρ Bernoulli for $i \neq 0$, and the second class particle Q starts at $Q(0) = 0$.

More generally, if two processes η and ω are in basic coupling and $\omega(0) \geq \eta(0)$ (by which we mean coordinatewise ordering $\omega_i(0) \geq \eta_i(0)$ for all i) then the ordering $\omega(t) \geq \eta(t)$ holds for all $0 \leq t < \infty$. The effect of the basic coupling is to give priority to the η particles over the $\omega - \eta$ particles. Consequently we can think of the ω-process as consisting of first class particles (the η particles) and second class particles (the $\omega - \eta$ particles).

Current. For $x \in \mathbb{Z}$ and $t > 0$, $J_x(t)$ stands for the net left-to-right particle current across the straight-line space-time path from $(1/2, 0)$ to $(x + 1/2, t)$. More precisely, $J_x(t) = J_x(t)^+ - J_x(t)^-$ where $J_x(t)^+$ is the number of particles that lie in $(-\infty, 0]$ at time 0 but lie in $[x+1, \infty)$ at time t, while $J_x(t)^-$ is the number of particles that lie in $[1, \infty)$ at time 0 and in $(-\infty, x]$ at time t. When more than one process (ω, η, etc) is considered in a coupling, the currents of the processes are denoted by $J_x^\omega(t)$, $J_x^\eta(t)$, etc.

5.2. Results

The average net rate at which particles in the stationary (p,q)-ASEP at density ρ move across a fixed edge $(i, i+1)$ is the *flux*

$$H(\rho) = E^\rho[J_0(t)] = (p-q)\rho(1-\rho). \tag{5.1}$$

This formula follows from the fact that this process $M(t)$ is a mean zero martingale:

$$M(t) = J_0(t) - \int_0^t \Big(p\mathbf{1}\{\eta_0(s) = 1,\, \eta_1(s) = 0\} - q\mathbf{1}\{\eta_1(s) = 1,\, \eta_0(s) = 0\} \Big)\, ds. \tag{5.2}$$

For the more general currents

$$E^\rho[J_x(t)] = tH(\rho) - x\rho \qquad (x \in \mathbb{Z}, t \geq 0) \tag{5.3}$$

as can be seen by noting that particles that crossed the edge $(0,1)$ either also crossed $(x, x+1)$ and contributed to $J_x(t)$ or did not.

The *characteristic speed* at density ρ is

$$V^\rho = H'(\rho) = (p-q)(1-2\rho). \tag{5.4}$$

The derivation of the fluctuation bounds for the current rests on several key identities which we collect in the next theorem.

THEOREM 5.1. *Let the second class particle start at the origin: $Q(0) = 0$. For any density $0 < \rho < 1$, $z \in \mathbb{Z}$ and $t > 0$ we have these formulas.*

$$\operatorname{Var}^\rho[J_z(t)] = \sum_{j\in\mathbb{Z}} |j - z| \operatorname{Cov}^\rho[\eta_j(t), \eta_0(0)], \tag{5.5}$$

$$\operatorname{Cov}^\rho[\eta_j(t),\, \eta_0(0)] = \rho(1-\rho)\mathbf{P}^\rho\{Q(t) = j\}, \tag{5.6}$$

and

$$\mathbf{E}^{\rho}[Q(t)] = V^{\rho}t. \tag{5.7}$$

Formulas (5.5) and (5.6) combine to give

$$\mathrm{Var}^{\rho}[J_z(t)] = \rho(1-\rho)\mathbf{E}^{\rho}|Q(t) - z|. \tag{5.8}$$

In particular, for the current across the characteristic,

$$\mathrm{Var}^{\rho}[J_{\lfloor V^{\rho}t\rfloor}(t)] = \rho(1-\rho)\mathbf{E}^{\rho}|Q(t) - \lfloor V^{\rho}t\rfloor|. \tag{5.9}$$

Thus to get variance bounds on the current, we derive moment bounds on the second class particle.

We now state the main result, the moment bounds on the second class particle. It is of interest to see how the bounds depend on the bias $\theta = p - q$ so we include that in the estimates.

THEOREM 5.2. *There exist constants $0 < c_0, C < \infty$ such that, for all $0 < \theta < 1/2$, $0 < \rho < 1$, $1 \le m < 3$, and $t \ge c_0\theta^{-4}$,*

$$\frac{1}{C}\theta^{2m/3}t^{2m/3} \le \mathbf{E}^{\rho}\big[\,|Q(t) - V^{\rho}t|^m\,\big] \le \frac{C}{3-m}\theta^{2m/3}t^{2m/3}. \tag{5.10}$$

For the upper bound the constants are fixed for all values of the parameters. For the lower bound both constants c_0, C depend on the density ρ.

As a corollary for $m = 1$, we obtain the bounds for the variance of the current seen by an observer traveling at the characteristic speed V^{ρ}: for $t \ge c_0(\rho)\theta^{-4}$,

$$C_1(\rho)\theta^{1/3}t^{2/3} \le \mathrm{Var}^{\rho}[J_{\lfloor V^{\rho}t\rfloor}(t)] \le C_2\theta^{1/3}t^{2/3}. \tag{5.11}$$

A distributional limit exists for the current for the case of the totally asymmetric simple exclusion process (TASEP). We state the result here. In TASEP particles march only to the right (say), and so $p = 1$ and $q = 0$.

THEOREM 5.3. [**FS06**] *In stationary TASEP, the following distributional convergence holds:*

$$\lim_{t\to\infty} P^{\rho}\left\{ \frac{J_{\lfloor V^{\rho}t\rfloor}(t) - \rho^2 t}{\rho^{2/3}(1-\rho)^{2/3}t^{1/3}} \le x \right\} = F_0(x) \tag{5.12}$$

The distribution function F_0 above is defined in [**FS06**] as $F_0(x) = (\partial/\partial x)(F_{\mathrm{GUE}}(x)g(x,0))$ where F_{GUE} is the Tracy-Widom GUE distribution and g a certain scaling function.

Theorem 5.3 will not be discussed further, and we turn to proofs of Theorem 5.1 and Theorem 5.2. In the next section we give partial proofs of the identities in Theorem 5.1. Section 5.4 describes a coupling that we use to control second class particles, and a random walk bound that comes in handy. The last two sections of this chapter prove the upper and lower bounds of Theorem 5.2.

5.3. Proofs for the identities

Let ω be a stationary exclusion process with i.i.d. Bernoulli(ρ) distributed occupations $\{\omega_i(t)\}$ at any fixed time t.

PROOF OF EQUATION (5.5). This is partly a hand-waiving proof. What is missing is justification for certain limits.

To approximate the infinite system with finite systems, for each $N \in \mathbb{N}$ let process ω^N have initial configuration

$$\omega_i^N(0) = \omega_i(0)\mathbf{1}_{\{-N\le i\le N\}}. \tag{5.13}$$

We assume that all these processes are coupled through common Poisson clocks. Let $J_z^N(t)$ denote the current in process ω^N.

Let $z(0) = 0$, $z(t) = z$, and introduce the counting variables

$$I_+^N(t) = \sum_{n > z(t)} \omega_n^N(t)\,, \quad I_-^N(t) = \sum_{n \le z(t)} \omega_n^N(t). \tag{5.14}$$

Then the current can be expressed as

$$J_z^N(t) = I_+^N(t) - I_+^N(0) = I_-^N(0) - I_-^N(t),$$

and its variance as

$$\begin{aligned}\operatorname{Var} J_z^N(t) &= \operatorname{Cov}\big(I_+^N(t) - I_+^N(0),\, I_-^N(0) - I_-^N(t)\big)\\ &= \operatorname{Cov}\big(I_+^N(t), I_-^N(0)) + \operatorname{Cov}(I_+^N(0), I_-^N(t))\\ &\quad - \operatorname{Cov}(I_+^N(0), I_-^N(0)) - \operatorname{Cov}(I_+^N(t), I_-^N(t)).\end{aligned}$$

Independence of initial occupation variables gives

$$\operatorname{Cov}(I_+^N(0), I_-^N(0)) = 0$$

and the identity above simplifies to

$$\begin{aligned}\operatorname{Var} J_z^N(t) &= \operatorname{Cov}\big(I_+^N(t), I_-^N(0)) + \operatorname{Cov}(I_+^N(0), I_-^N(t)) - \operatorname{Cov}(I_+^N(t), I_-^N(t))\\ &= \sum_{k \le 0,\, m > z} \operatorname{Cov}[\omega_m^N(t), \omega_k^N(0)]\\ &\quad + \sum_{k \le z,\, m > 0} \operatorname{Cov}[\omega_k^N(t), \omega_m^N(0)] - \operatorname{Cov}(I_+^N(t), I_-^N(t)).\end{aligned} \tag{5.15}$$

In the $N \to \infty$ limit variables $\omega_i^N(t)$ converge (a.s. and in L^2) to the i.i.d. occupation variables $\omega_i(t)$ of the stationary process. It follows from the graphical construction that on a fixed time interval covariances can be bounded exponentially, uniformly over N: for a fixed $0 < t < \infty$,

$$|\operatorname{Cov}[\omega_m^N(t), \omega_k^N(s)]| \le Ce^{-c_1|m-k|} \quad \text{for } s \in [0, t].$$

Hence in the limit the last covariance in (5.15) vanishes. Furthermore, $J_z^N(t) \to J_z(t)$ similarly, so in the limit we get

$$\begin{aligned}\operatorname{Var} J_z(t) &= \sum_{k \le 0,\, m > z} \operatorname{Cov}[\omega_m(t), \omega_k(0)] + \sum_{k \le z,\, m > 0} \operatorname{Cov}[\omega_k(t), \omega_m(0)]\\ &= \sum_{n \in \mathbb{Z}} |n - z| \operatorname{Cov}[\omega_n(t), \omega_0(0)].\end{aligned} \tag{5.16}$$

This proves equation (5.5). □

Proof of equation (5.6). This is a straight-forward calculation.

$$\begin{aligned}&\operatorname{Cov}^\rho[\omega_j(t), \omega_0(0)] = E^\rho[\omega_j(t)\omega_0(0)] - \rho^2 = \rho E^\rho[\omega_j(t) \,|\, \omega_0(0) = 1] - \rho E^\rho[\omega_j(t)]\\ &= \rho\Big(E^\rho[\omega_j(t) \,|\, \omega_0(0) = 1] - \rho E^\rho[\omega_j(t) \,|\, \omega_0(0) = 1] - (1 - \rho) E^\rho[\omega_j(t) \,|\, \omega_0(0) = 0]\Big)\\ &= \rho(1 - \rho)\Big(E^\rho[\omega_j(t) \,|\, \omega_0(0) = 1] - E^\rho[\omega_j(t) \,|\, \omega_0(0) = 0]\Big)\\ &= \rho(1 - \rho)\big(\mathbf{E}^\rho[\omega_j^+(t)] - \mathbf{E}^\rho[\omega_j(t)]\big) = \rho(1 - \rho)\mathbf{E}^\rho[\omega_j^+(t) - \omega_j(t)]\\ &= \rho(1 - \rho)\mathbf{P}^\rho[Q(t) = j].\end{aligned}$$

□

Proof of equation (5.7). Let again ω^N be the finite process with initial condition (5.13). Let $I^N = \sum_i \omega_i^N(t)$ be the number of particles in the process ω^N. I^N is a Binomial$(2N+1,\rho)$ random variable. For $0<\rho<1$

$$\begin{aligned}\frac{d}{d\rho}E[J_z^N(t)] &= \frac{d}{d\rho}\sum_{m=0}^{2N+1}\binom{2N+1}{m}\rho^m(1-\rho)^{2N+1-m}E[J_z^N(t)|I^N=m]\\ &= \sum_{m=0}^{2N+1}P(I^N=m)\Big(\frac{m}{\rho}-\frac{2N+1-m}{1-\rho}\Big)E[J_z^N(t)|I^N=m]\\ &= \frac{1}{\rho(1-\rho)}E\Big[J_z^N(t)\big(I^N-(2N+1)\rho\big)\Big]\\ &= \frac{1}{\rho(1-\rho)}\operatorname{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_-^N(0)+I_+^N(0)\big]\\ &= \frac{1}{\rho(1-\rho)}\Big(\operatorname{Cov}\big[I_+^N(t)\,,\,I_-^N(0)\big]+\operatorname{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_+^N(0)\big]\Big).\end{aligned} \tag{5.17}$$

The last equality used $\operatorname{Cov}[I_+^N(0), I_-^N(0)] = 0$ that comes from the i.i.d. distribution of initial occupations. The first covariance on line (5.17) write directly as

$$\operatorname{Cov}\big[I_+^N(t)\,,\,I_-^N(0)\big] = \sum_{k\le 0,\, m>z}\operatorname{Cov}[\omega_m^N(t),\omega_k^N(0)].$$

The second covariance on line (5.17) write as

$$\begin{aligned}\operatorname{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_+^N(0)\big] &= \operatorname{Cov}\big[I_-^N(0)-I_-^N(t)\,,\,I_+^N(0)\big]\\ &= -\operatorname{Cov}\big[I_-^N(t)\,,\,I_+^N(0)\big] = -\sum_{k\le z,\, m>0}\operatorname{Cov}[\omega_k^N(t),\omega_m^N(0)].\end{aligned}$$

Inserting these back on line (5.17) gives

$$\frac{d}{d\rho}E[J_z^N(t)] = \frac{1}{\rho(1-\rho)}\Big(\sum_{k\le 0,\, m>z}\operatorname{Cov}[\omega_m^N(t),\omega_k^N(0)] - \sum_{k\le z,\, m>0}\operatorname{Cov}[\omega_k^N(t),\omega_m^N(0)]\Big).$$

Compared to line (5.15) we have the difference instead of the sum. Integrate over the density ρ and take $N\to\infty$ as was taken from (5.15) to (5.16) to obtain

$$\begin{aligned}E^\rho[J_z(t)] - E^\lambda[J_z(t)] &= \int_\lambda^\rho \frac{1}{\theta(1-\theta)}\sum_{j\in\mathbb{Z}}(j-z)\operatorname{Cov}^\theta[\omega_j(t),\,\omega_0(0)]\,d\theta\\ &= \int_\lambda^\rho (\mathbf{E}^\theta[Q(t)]-z)\,d\theta\end{aligned}$$

for $0<\lambda<\rho<1$. Couplings show the continuity of these expectations:

$$E^\lambda[J_z(t)]\to E^\rho[J_z(t)] \quad\text{and}\quad \mathbf{E}^\lambda[Q(t)]\to\mathbf{E}^\rho[Q(t)] \quad \text{as } \lambda\to\rho \text{ in } (0,1). \tag{5.18}$$

Thus the identity above can be differentiated in ρ. With $z=0$ and via (5.1) identity (5.7) follows. ☐

5.4. A coupling and a random walk bound

As observed in (5.7) the mean speed of the second class particle in a density-ρ ASEP is $H'(\rho)$. Thus by the concavity of H a defect travels on average slower in a denser system (recall that we assume $p>q$ throughout). However, the basic coupling does not respect this, except in the totally asymmetric ($p=1, q=0$) case. To see this, consider two pairs of processes (ω^+,ω) and (η^+,η) such that both pairs have one discrepancy: $\omega^+(t) = \omega(t) + \delta_{Q^\omega(t)}$ and $\eta^+(t) = \eta(t) + \delta_{Q^\eta(t)}$. Assume that $\omega(t) \geq \eta(t)$. In basic coupling the jump from state

$$\begin{bmatrix} \omega_i^+ & \omega_{i+1}^+ \\ \omega_i & \omega_{i+1} \\ \eta_i^+ & \eta_{i+1}^+ \\ \eta_i & \eta_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to state} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

happens at rate q and results in $Q^\omega = i + 1 > i = Q^\eta$.

In this section we construct a different coupling that combines the basic coupling with auxiliary clocks for second class particles. The idea is to think of a single "special" second class particle as performing a random walk on the process of $\omega - \eta$ second class particles. This coupling preserves the expected ordering of the special second class particles, hence it can be regarded as a form of *microscopic concavity*.

This theorem summarizes the outcome.

Theorem 5.4. *Assume given two initial configurations $\{\zeta_i(0)\}$ and $\{\xi_i(0)\}$ and two not necessarily distinct positions $Q^\zeta(0)$ and $Q^\xi(0)$ on $\mathbb{Z}$. Suppose the coordinatewise ordering $\zeta(0) \geq \xi(0)$ holds, $Q^\zeta(0) \leq Q^\xi(0)$, and $\zeta_i(0) = \xi_i(0) + 1$ for $i \in \{Q^\zeta(0), Q^\xi(0)\}$. Define the configuration $\zeta^-(0) = \zeta(0) - \delta_{Q^\zeta(0)}$.*

Then there exists a coupling of processes

$$(\zeta^-(t), Q^\zeta(t), \xi(t), Q^\xi(t))_{t\geq 0}$$

with initial state $(\zeta^-(0), Q^\zeta(0), \xi(0), Q^\xi(0))$ as described in the previous paragraph, such that both pairs (ζ^-, Q^ζ) and (ξ, Q^ξ) are (p, q)-ASEP's with a second class particle, and $Q^\zeta(t) \leq Q^\xi(t)$ for all $t \geq 0$.

To begin the construction, put two exclusion processes ζ and ξ in basic coupling, obeying Poisson clocks $\{N^{i\to i\pm 1}\}$. They are ordered so that $\zeta \geq \xi$. The $\zeta - \xi$ second class particles are labeled in increasing order $\cdots < X_{m-1}(t) < X_m(t) < X_{m+1}(t) < \cdots$. We assume there is at least one such second class particle, but beyond that we make no assumption about their number. Thus there is some finite or infinite subinterval $I \subseteq \mathbb{Z}$ of indices such that the positions of the $\zeta - \xi$ second class particles are given by $\{X_m(t) : m \in I\}$.

We introduce two dynamically evolving labels $a(t)$, $b(t) \in I$ in such a manner that $X_{a(t)}(t)$ is the position of a second class antiparticle in the ζ-process, $X_{b(t)}(t)$ is the position of a second class particle in the ξ-process, and the ordering

$$X_{a(t)}(t) \leq X_{b(t)}(t) \tag{5.19}$$

is preserved by the dynamics.

The labels $a(t)$, $b(t)$ are allowed to jump from m to $m \pm 1$ only when particle $X_{m\pm 1}$ is adjacent to X_m. The labels do not take jump commands from the Poisson clocks $\{N^{i\to i\pm 1}\}$ that govern (ξ, ζ). Instead, the directed edges $(i, i + 1)$ and $(i, i - 1)$ are given another collection of independent Poisson clocks so that the following jump rates are realized.

(i) If $a = b$ and $X_{a+1} = X_a + 1$ then

$$(a, b) \text{ jumps to } \begin{cases} (a, b + 1) & \text{with rate } p - q \\ (a + 1, b + 1) & \text{with rate } q. \end{cases}$$

(ii) If $a = b$ and $X_{a-1} = X_a - 1$ then

$$(a, b) \text{ jumps to } \begin{cases} (a - 1, b) & \text{with rate } p - q \\ (a - 1, b - 1) & \text{with rate } q. \end{cases}$$

(iii) If $a \neq b$ then a and b jump independently with these rates:

$$a \text{ jumps to } \begin{cases} a+1 & \text{with rate } q \text{ if } X_{a+1} = X_a + 1 \\ a-1 & \text{with rate } p \text{ if } X_{a-1} = X_a - 1; \end{cases}$$

$$b \text{ jumps to } \begin{cases} b+1 & \text{with rate } p \text{ if } X_{b+1} = X_b + 1 \\ b-1 & \text{with rate } q \text{ if } X_{b-1} = X_b - 1. \end{cases}$$

Let us emphasize that the pair process (ξ, ζ) is still governed by the old clocks $\{N^{i\to i\pm 1}\}$ in the basic coupling. The new clocks on edge $\{i, i+1\}$ that realize rules (i)–(iii) are not observed except when sites $\{i, i+1\}$ are both occupied by X-particles and at least one of X_a or X_b lies in $\{i, i+1\}$.

First note that if initially $a(0) \leq b(0)$ then jumps (i)–(iii) preserve the inequality $a(t) \leq b(t)$ which gives (5.19). (Since the jumps in point (iii) happen independently, there cannot be two simultaneous jumps. So it is not possible for a and b to cross each other with a $(a, b) \to (a+1, b-1)$ move.)

Define processes $\zeta^-(t) = \zeta(t) - \delta_{X_{a(t)}(t)}$ and $\xi^+(t) = \xi(t) + \delta_{X_{b(t)}(t)}$. In other words, to produce ζ^- remove particle X_a from ζ, and to produce ξ^+ add particle X_b to ξ. The second key point is that, even though these new processes are no longer defined by the standard graphical construction, distributionwise they are still ASEP's with second class particles. We argue this point for (ζ^-, X_a) and leave the argument for (ξ, X_b) to the reader.

Lemma 5.5. *The pair (ζ^-, X_a) is a (p, q)-ASEP with a second class particle.*

Proof. We check that the jump rates for the process (ζ^-, X_a), produced by the combined effect of the basic coupling with clocks $\{N^{i\to i\pm 1}\}$ and the new clocks, are the same jump rates that result from defining an (ASEP, second class particle) pair in terms of the graphical construction.

To have notation for the possible jumps, let 0 denote an empty site, 1 a ζ^--particle, and 2 particle X_a. Consider a fixed pair $(i, i+1)$ of sites and write xy with $x, y \in \{0, 1, 2, \}$ for the contents of sites $(i, i+1)$ before and after the jump. Then here are the possible moves across the edge $\{i, i+1\}$, and the rates that these moves would have in the basic coupling.

Type 1	$10 \longrightarrow 01$	with rate p
	$01 \longrightarrow 10$	with rate q
Type 2	$20 \longrightarrow 02$	with rate p
	$02 \longrightarrow 20$	with rate q
Type 3	$12 \longrightarrow 21$	with rate p
	$21 \longrightarrow 12$	with rate q

Our task is to check that the construction of (ζ^-, X_a) actually realizes these rates.

Jumps of types 1 and 2 are prompted by the clocks $\{N^{i\to i\pm 1}\}$ of the graphical construction of (ξ, ζ), and hence have the correct rates listed above.

Jumps of type 3 occur in two distinct ways.

(Type 3.1) First there can be a ξ-particle next to X_a, and then the rates shown above are again realized by the clocks $\{N^{i\to i\pm 1}\}$ because in the basic coupling the ξ-particles have priority over the X-particles.

(Type 3.2) The other alternative is that both sites $\{i, i+1\}$ are occupied by X-particles and one of them is X_a. The clocks $\{N^{i\to i\pm 1}\}$ cannot interchange the X-particles across the edge $\{i, i+1\}$ because in the (ξ, ζ)-graphical construction these are lower priority ζ-particles that do not jump on top of each other. The otherwise missing jumps are now supplied by the "new" clocks that govern the jumps described in rules (i)–(iii).

Combining (i)–(iii) we can read that if $X_a = i+1$ and $X_{a-1} = i$, then a jumps to $a-1$ with rate p. This is the first case of type 3 jumps above, corresponding to a ζ^--particle moving from i to $i+1$ with rate p, and the second class particle X_a yielding. On the other hand, if $X_a = i$ and $X_{a+1} = i+1$ then a jumps to $a+1$ with rate q. This is the second case in type 3, corresponding to a ζ^--particle moving from $i+1$ to i with rate q and exchanging places with the second class particle X_a.

We have verified that the process (ζ^-, X_a) operates with the correct rates.

To argue from the rates to the correct distribution of the process, we can make use of the process (ζ^-, ζ). The processes (ζ^-, X_a) and (ζ^-, ζ) determine each other uniquely. The virtue of (ζ^-, ζ) is that it has a compact state space and only nearest-neighbor jumps with bounded rates. Hence by the basic theory of semigroups and generators of particle systems as developed in [**Lig85**], given the initial configuration, the distribution of the process is uniquely determined by the action of the generator on local functions. Thus it suffices to check that individual jumps have the correct rates across each edge $\{i, i+1\}$. This is exactly what we did above in the language of (ζ^-, X_a). □

Similar argument shows that (ξ, X_b) is a (p,q)-ASEP with a second class particle. To prove Theorem 5.4 take $Q^\zeta = X_a$ and $Q^\xi = X_b$. This gives the coupling whose existence is claimed in the theorem.

To conclude, let us observe that the four processes $(\xi, \xi^+, \zeta^-, \zeta)$ are not in basic coupling. For example, the jump from state

$$\begin{bmatrix} \zeta_i & \zeta_{i+1} \\ \zeta_i^- & \zeta_{i+1}^- \\ \xi_i^+ & \xi_{i+1}^+ \\ \xi_i & \xi_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to state} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

happens at rate q (second case of rule (i)), while in basic coupling this move is impossible.

As the second point of this section we prove a random walk estimate. Let $Z(t)$ be a continuous-time nearest-neighbor random walk on state-space $S \subseteq \mathbb{Z}$ that contains $\mathbb{Z}_- = \{\dots, -2, -1, 0\}$. Initially $Z(0) = 0$. Z attempts to jump from x to $x+1$ with rate p for $x \le -1$, and from x to $x-1$ with rate q for $x \le 0$. Assume $p > q = 1-p$ and let $\theta = p - q$. The rates on $S \smallsetminus \mathbb{Z}_-$ need not be specified.

Whether jumps are permitted or not is determined by a fixed environment expressed in terms of $\{0,1\}$-valued functions $\{u(x,t) : x \in S,\ 0 \le t < \infty\}$. A jump across edge $\{x-1, x\}$ in either direction is permitted at time t if $u(x,t) = 1$, otherwise not. In other words, $u(x,t)$ is the indicator of the event that edge $\{x-1, x\}$ is open at time t.

Assumption. Assume that for all $x \in S$ and $T < \infty$, $u(x,t)$ flips between 0 and 1 only finitely many times during $0 \le t \le T$. Assume for convenience right-continuity: $u(x, t+) = u(x,t)$.

LEMMA 5.6. *For all $t \ge 0$ and $k \ge 0$,*

$$P\{Z(t) \le -k\} \le e^{-2\theta k}.$$

This bound holds for any fixed environment $\{u(x,t)\}$ subject to the assumption above.

PROOF. Let $Y(t)$ be a walk that operates exactly as $Z(t)$ on $\mathbb{Z}_-$ but is restricted to remain in $\mathbb{Z}_-$ by setting the rate of jumping from 0 to 1 to zero. Give $Y(t)$ geometric initial distribution

$$P\{Y(0) = -j\} = \pi(j) \equiv \Big(1 - \frac{q}{p}\Big)\Big(\frac{q}{p}\Big)^j \quad \text{for } j \ge 0.$$

The initial points satisfy $Y(0) \le Z(0)$ a.s. Couple the walks through Poisson clocks so that the inequality $Y(t) \le Z(t)$ is preserved for all time $0 \le t < \infty$.

Without the inhomogeneous environment $Y(t)$ would be a stationary, reversible birth and death process. We argue that even with the environment the time marginals $Y(t)$ still have distribution π. This suffices for the conclusion, for then

$$P\{Z(t) \le -k\} \le P\{Y(t) \le -k\} = (q/p)^k = \exp\bigl(k \log \tfrac{1-\theta}{1+\theta}\bigr) \le e^{-2\theta k}.$$

To justify the claim about $Y(t)$, consider approximating processes $Y^{(m)}(t)$, $m \in \mathbb{N}$, with the same initial value $Y^{(m)}(0) = Y(0)$. $Y^{(m)}(t)$ evolves so that the environments $\{u(x,t)\}$ restrict its motion only on edges $\{x-1, x\}$ for $-m+1 \le x \le 0$. In other words, for walk $Y^{(m)}(t)$ we set $u(x,t) \equiv 1$ for $x \le -m$ and $0 \le t < \infty$. We couple the walks together so that $Y(t) = Y^{(m)}(t)$ until the first time one of the walks exits the interval $\{-m+1, \dots, 0\}$.

Fixing m for a moment, let $0 = s_0 < s_1 < s_2 < s_3 < \dots$ be a partition of the time axis so that $s_j \nearrow \infty$ and the environments $\{u(x,t) : -m < x \le 0\}$ are constant on each interval $t \in [s_i, s_{i+1})$. Then on each time interval $[s_i, s_{i+1})$ $Y^{(m)}(t)$ is a continuous time Markov chain with time-homogeneous jump rates

$$c(x, x+1) = \begin{cases} pu(x+1, s_i), & -m \le x \le 0 \\ p, & x \le -m-1 \end{cases}$$

and

$$c(x, x-1) = \begin{cases} qu(x, s_i), & -m+1 \le x \le 0 \\ q, & x \le -m. \end{cases}$$

One can check that detailed balance $\pi(x)c(x, x+1) = \pi(x+1)c(x+1, x)$ holds for all $x \le -1$. Thus π is a reversible measure for walk $Y^{(m)}(t)$ on each time interval $[s_i, s_{i+1})$, and we conclude that $Y^{(m)}(t)$ has distribution π for all $0 \le t < \infty$.

The coupling ensures that $Y^{(m)}(t) \to Y(t)$ almost surely as $m \to \infty$, and consequently also $Y(t)$ has distribution π for all $0 \le t < \infty$. ☐

5.5. Proof of the upper bound for second class particle moments

Abbreviate

$$\Psi(t) = \mathbf{E}^\rho |Q(t) - V^\rho t|. \tag{5.20}$$

LEMMA 5.7. *Let $B \in (0, \infty)$. Then there exists a numerical constant $C \in (0, \infty)$ and another constant $c_1(B) \in (0, \infty)$ such that, for all densities $0 < \rho < 1$, $u \ge 1$, $0 < \theta < 1/2$, and $t \ge c_1(B)\theta^{-1}$,*

$$\mathbf{P}^\rho\{Q(t) \ge V^\rho t + u\} \le \begin{cases} C\theta^2\Bigl(\dfrac{t^2}{u^4}\Psi(t) + \dfrac{t^2}{u^3}\Bigr) + e^{-u^2/Ct}, & B\theta^{2/3}t^{2/3} \le u \le 20t/3 \\ e^{-u/C}, & u \ge 20t/3. \end{cases} \tag{5.21}$$

PROOF. First we get an easy case out of the way.

Case 1. $u \ge 5\rho\theta t$.

This comes from an exponential Chebyshev argument. Let Z_t be a nearest-neighbor random walk with rates $p = (1+\theta)/2$ to the right and $q = (1-\theta)/2$ to the left. For $\alpha \in (0,1]$, using

$$\frac{e^\alpha + e^{-\alpha}}{2} \le 1 + \alpha^2 \quad \text{and} \quad \frac{e^\alpha - e^{-\alpha}}{2} \le \alpha + \alpha^2,$$

we get

$$\begin{aligned} \mathbf{E}[e^{\alpha Z_t}] &= \exp\Bigl(-t + t\frac{e^\alpha + e^{-\alpha}}{2} + \theta t \frac{e^\alpha - e^{-\alpha}}{2}\Bigr) \\ &\le \exp\bigl(2\alpha^2 t + \alpha\theta t\bigr). \end{aligned} \tag{5.22}$$

We have the stochastic domination $Z_t \ge Q(t)$ because no matter what the environment next to $Q(t)$ is, $Q(t)$ has a weaker right drift than Z_t. Then, since $V^\rho = \theta(1-2\rho)$ and $2\rho\theta t \le 2u/5$,

$$\begin{aligned} \mathbf{P}^\rho\{Q(t) \ge V^\rho t + u\} &\le P\{Z_t \ge \theta t + \tfrac{3}{5}u\} \\ &\le \exp\bigl(-\tfrac{3}{5}\alpha u + 2\alpha^2 t\bigr) \\ &\le \begin{cases} \exp(-\frac{9u^2}{200t}) & u \le 20t/3 \\ \exp(-3u/10) & u > 20t/3. \end{cases} \end{aligned} \tag{5.23}$$

In the last inequality above choose $\alpha = 1 \wedge \frac{3u}{20t}$. Note that $20t/3 > 5\rho\theta t$.

It remains to consider this range of u:

Case 2. $B\theta^{2/3}t^{2/3} \le u \le 5\rho\theta t$.

By an adjustment of the constant C we can assume that u is a positive integer. Fix a density $0 < \rho < 1$ and an auxiliary density

$$\lambda = \rho - \frac{u}{10\theta t}. \tag{5.24}$$

Start with the basic coupling of three exclusion processes $\omega \ge \omega^- \ge \eta$ with this initial set-up:

(a) Initially $\{\omega_i(0) : i \ne 0\}$ are i.i.d. Bernoulli(ρ) distributed and $\omega_0(0) = 1$.

(b) Initially $\omega^-(0) = \omega(0) - \delta_0$.

(c) Initially variables $\{\eta_i(0) : i \ne 0\}$ are i.i.d. Bernoulli(λ) and $\eta_0(0) = 0$. The coupling of the initial occupations is such that $\omega_i(0) \ge \eta_i(0)$ for all $i \ne 0$.

Recall that basic coupling meant that these processes obey common Poisson clocks.

Let $Q(t)$ be the position of the single second class particle between $\omega(t)$ and $\omega^-(t)$, initially at the origin. Let $\{X_i(t) : i \in \mathbb{Z}\}$ be the positions of the $\omega - \eta$ second class particles, initially labeled so that

$$\cdots < X_{-2}(0) < X_{-1}(0) < X_0(0) = 0 < X_1(0) < X_2(0) < \cdots$$

These second class particles preserve their labels in the dynamics and stay ordered. Thus the $\omega(t)$ configuration consists of first class particles (the $\eta(t)$ process) and second class particles (the $X_j(t)$'s). $\mathbf{P}$ denotes the probability measure under which all these coupled processes live. Note that the marginal distribution of (ω, ω^-, Q) under $\mathbf{P}$ is exactly as it would be under $\mathbf{P}^\rho$.

For $x \in \mathbb{Z}$, $J_x^\omega(t)$ is the net current in the ω-process between space-time positions $(1/2, 0)$ and $(x+1/2, t)$. Similarly $J_x^\eta(t)$ in the η-process, and $J_x^{\omega-\eta}(t)$ is the net current of second class particles. Current in the ω-process is a sum of the first class particle current and the second class particle current:

$$J_x^\omega(t) = J_x^\eta(t) + J_x^{\omega-\eta}(t). \tag{5.25}$$

$Q(t) \in \{X_j(t)\}$ for all time because the basic coupling preserves the ordering $\omega^-(t) \ge \eta(t)$. Define the label $m_Q(t)$ by $Q(t) = X_{m_Q(t)}(t)$ with initial value $m_Q(0) = 0$.

LEMMA 5.8. *For all $t \geq 0$ and $k \geq 0$,*

$$\mathbf{P}\{m_Q(t) \geq k\} \leq e^{-2\theta k}.$$

PROOF OF LEMMA 5.8. In the basic coupling the label $m_Q(t)$ evolves as follows. When X_{m_Q-1} is adjacent to X_{m_Q}, m_Q jumps down by one at rate p. And when X_{m_Q+1} is adjacent to X_{m_Q}, m_Q jumps up by one at rate q. When X_{m_Q} has no X-particle in either neighboring site, the label m_Q cannot jump. Thus the situation is like that in Lemma 5.6 (with a reversal of lattice directions) with environment given by the adjacency of X-particles: $u(m,t) = \mathbf{1}\{X_m(t) = X_{m-1}(t)+1\}$. However, the basic coupling mixes together the evolution of the environment and the walk m_Q, so the environment is not specified in advance as required by Lemma 5.6.

We can get around this difficulty by imagining an alternative but distributionally equivalent construction for the joint process (η, ω^-, ω). Let (η, ω) obey basic coupling with the given Poisson clocks $\{N^{x\to x\pm 1}\}$ attached to directed edges $(x, x\pm 1)$. Divide the $\omega - \eta$ particles further into class II consisting of the particles $\omega^- - \eta$ and class III that consists only of the single particle $\omega - \omega^- = \delta_Q$. Let class II have priority over class III. Introduce another independent set of Poisson clocks $\{\widetilde{N}^{x\to x\pm 1}\}$, also attached to directed edges $(x, x\pm 1)$ of the space $\mathbb{Z}$ where particles move. Let clocks $\{\widetilde{N}^{x\to x\pm 1}\}$ govern the exchanges between classes II and III. In other words, for each edge $\{x, x+1\}$ clocks $\widetilde{N}^{x\to x+1}$ and $\widetilde{N}^{x+1\to x}$ are observed if sites $\{x, x+1\}$ are both occupied by $\omega - \eta$ particles. All other jumps are prompted by the original clocks.

The rates for individual jumps are the same in this alternative construction as in the earlier one where all processes were together in basic coupling. Thus the same distribution for the process (η, ω^-, ω) is created.

To apply Lemma 5.6 perform the construction in two steps. First construct the process (η, ω) for all time. This determines the environment $u(m,t) = \mathbf{1}\{X_m(t) = X_{m-1}(t)+1\}$. Then run the dynamics of classes II and III in this environment. Now Lemma 5.6 gives the bound for m_Q. ☐

Let u be a positive integer and

$$k = \left\lfloor \frac{u^2}{20\theta t} \right\rfloor - 3. \tag{5.26}$$

By assuming $t \geq C(B)\theta^{-1}$ we guarantee that

$$u \geq 1 \quad \text{and} \quad \frac{u^2}{40\theta t} \geq \frac{B^2\theta^{1/3}t^{1/3}}{40} \geq 4.$$

Then

$$k \geq \frac{u^2}{40\theta t} \geq 4. \tag{5.27}$$

We begin a series of inequalities.

$$\begin{aligned} &\mathbf{P}\{Q(t) \geq V^\rho t + u\} \\ &\quad \leq \mathbf{P}\{m_Q(t) \geq k\} + \mathbf{P}\{J^\omega_{\lfloor V^\rho t\rfloor + u}(t) \;-\; J^\eta_{\lfloor V^\rho t\rfloor + u}(t) > -k\}. \end{aligned} \tag{5.28}$$

To explain the inequality above, if $Q(t) \geq V^\rho t + u$ and $m_Q(t) < k$ then $X_k(t) > \lfloor V^\rho t\rfloor + u$. This puts the bound

$$J^{\omega-\eta}_{\lfloor V^\rho t\rfloor + u}(t) > -k$$

on the second class particle current, because at most particles $X_1, \ldots, X_{k-1}$ could have made a negative contribution to this current.

Lemma 5.8 takes care of the first probability on line (5.28). We work on the second probability on line (5.28).

Here is a simple observation that will be used repeatedly. Process ω can be coupled with a stationary density-ρ process $\omega^{(\rho)}$ so that the coupled pair $(\omega, \omega^{(\rho)})$ has at most 1 discrepancy. In this coupling

$$|J_x^{\omega}(t) - J_x^{\omega^{(\rho)}}(t)| \le 1. \tag{5.29}$$

This way we can use computations for stationary processes at the expense of small errors.

Recall that $V^\rho = H'(\rho)$. Let c_1 below be a constant that absorbs the errors from using means of stationary processes and from ignoring integer parts. It satisfies $|c_1| \le 3$.

$$\begin{aligned}
\mathbf{E}J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) - \mathbf{E}J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) &= tH(\rho) - (H'(\rho)t+u)\rho - tH(\lambda) + (H'(\rho)t+u)\lambda + c_1 \\
&= -\tfrac{1}{2}tH''(\rho)(\rho-\lambda)^2 - u(\rho-\lambda) + c_1 \\
&= t\theta(\rho-\lambda)^2 - u(\rho-\lambda) + c_1. \\
&= t\theta(\rho-\lambda)^2 - u(\rho-\lambda) + c_1 + k - k \\
&\le \frac{u^2}{100t\theta} - \frac{u^2}{10t\theta} + \frac{u^2}{20t\theta} - k \\
&= -\frac{u^2}{25t\theta} - k. \qquad (5.30)
\end{aligned}$$

The -3 in the definition (5.26) of k absorbed c_1 above.

Let $\overline{X} = X - EX$ denote a centered random variable. Continuing with the second probability from line (5.28):

$$\begin{aligned}
&\mathbf{P}\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) > -k\} \\
&\le \mathbf{P}\Big\{\overline{J}^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; \overline{J}^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) \ge \frac{u^2}{25t\theta}\Big\} \\
&\le \frac{C\theta^2t^2}{u^4}\mathbf{Var}\Big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\Big\} \\
&\le \frac{C\theta^2t^2}{u^4}\Big(\mathbf{Var}\big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t)\big\} \;+\; \mathbf{Var}\big\{J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\big\}\Big). \qquad (5.31)
\end{aligned}$$

C is a numerical constant that can change from line to line but is independent of all the parameters.

We develop bounds on the variances above, first for J^ω. Pass to the stationary density-ρ process via (5.29) and apply (5.8):

$$\begin{aligned}
\mathbf{Var}\big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t)\big\} &\le 2\,\mathrm{Var}^\rho\big\{J_{\lfloor V^\rho t\rfloor+u}(t)\big\} + 2 \\
&= 2\rho(1-\rho)\mathbf{E}\big|Q(t) - \lfloor V^\rho t\rfloor - u\big| + 2 \\
&\le \mathbf{E}|Q(t) - V^\rho t\,| + u + 3 \\
&\le \Psi(t) + 4u. \qquad (5.32)
\end{aligned}$$

Let Var^λ denote variance in the stationary density-λ process and let $Q^\eta(t)$ denote the position of a second class particle added to a process η.

$$\begin{aligned}
\mathbf{Var}\big\{J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\big\} &\le 2\,\mathrm{Var}^\lambda\big\{J_{\lfloor V^\rho t\rfloor+u}(t)\big\} + 2 \\
&\le \mathbf{E}^\lambda\big|Q^\eta(t) - \lfloor V^\rho t\rfloor - u\big| + 2 \\
&\le \mathbf{E}^\lambda|Q^\eta(t) - V^\rho t\,| + 4u
\end{aligned}$$

Introduce process $(\zeta^-(t), Q^\zeta(t), \eta(t), Q^\eta(t))_{t\ge 0}$ coupled as in Theorem 5.4, where ζ starts with Bernoulli(ρ) occupations away from the origin and initially $Q^\zeta(0) = Q^\eta(0) = 0$. Below apply the triangle inequality and use inequality $Q^\zeta(t) \le Q^\eta(t)$ from Theorem 5.4. Thus continuing from above:

$$\begin{aligned} &= \mathbf{E}|Q^\eta(t) - Q^\zeta(t) + Q^\zeta(t) - V^\rho t\,| + 4u \\ &\le \mathbf{E}\big\{Q^\eta(t) - Q^\zeta(t)\big\} + \mathbf{E}|Q^\zeta(t) - V^\rho t\,| + 4u \\ &= V^\lambda t - V^\rho t + \Psi(t) + 4u \\ &= 2\theta t(\rho - \lambda) + \Psi(t) + 4u \\ &= \Psi(t) + 5u. \end{aligned} \tag{5.33}$$

Marginally the process (ζ, Q^ζ) is the same as the process (ω, Q) in the coupling of this section, hence the appearance of $\Psi(t)$ above. Then we used (5.7) for the expectations of the second class particles and the choice (5.24) of λ.

Insert bounds (5.32) and (5.33) into (5.31) to get

$$\mathbf{P}\{J^\omega_{\lfloor V^\rho t\rfloor + u}(t) \;-\; J^\eta_{\lfloor V^\rho t\rfloor + u}(t) > -k\} \le C\theta^2\Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big). \tag{5.34}$$

Insert (5.27) and (5.34) into line (5.28) to get

$$\mathbf{P}\{Q(t) \le V^\rho t - u\} \le C\theta^2\Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big) + e^{-u^2/20t} \tag{5.35}$$

and we have verified (5.21) for **Case 2**.

Combining (5.35) and (5.23) gives the conclusion of Lemma 5.7. □

Next we extend the bound to both tails.

LEMMA 5.9. *Let $B \in (0,\infty)$. Then there exists a numerical constant $C \in (0,\infty)$ and another constant $c_0(B) \in (0,\infty)$ such that, for all densities $0 < \rho < 1$ and $t \ge c_0(B)\theta^{-1}$,*

$$\begin{aligned} &\mathbf{P}^\rho\{\,|Q(t) - V^\rho t| \ge u\} \\ &\le \begin{cases} C\theta^2\Big(\dfrac{t^2}{u^4}\Psi(t) + \dfrac{t^2}{u^3}\Big) + 2e^{-u^2/Ct}, & B\theta^{2/3}t^{2/3} \le u \le 20t/3 \\ 2e^{-u/C}, & u \ge 20t/3. \end{cases} \end{aligned} \tag{5.36}$$

PROOF. The corresponding lower tail bound is obtained from (5.21) by a particle-hole interchange followed by a reflection of the lattice. For details we refer to Lemma 5.3 in [**BS09a**]. □

PROOF OF THE UPPER BOUND OF THEOREM 5.2. We integrate (5.36) to get the bound (5.10) on the moments of the second class particle. First for $m = 1$.

$$\begin{aligned} \Psi(t) &= \int_0^\infty \mathbf{P}^\rho\{\,|Q(t) - V^\rho t| \ge u\}\,du \\ &\le B\theta^{2/3}t^{2/3} + C\theta^2\int_{B\theta^{2/3}t^{2/3}}^\infty \Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big)\,du \\ &\qquad + 2\int_{B\theta^{2/3}t^{2/3}}^\infty e^{-u^2/Ct}\,du + 2\int_{20t/3}^\infty e^{-u/C}\,du \\ &\le \frac{C}{3B^3}\Psi(t) + \Big(B + \frac{C}{2B^2}\Big)\theta^{2/3}t^{2/3} + \frac{C_1(B)t^{1/3}}{\theta^{2/3}}e^{-\theta^{4/3}t^{1/3}/C_1(B)} + 2Ce^{-t/C}. \end{aligned}$$

$C_1(B)$ is a new constant that depends on B. Set $B = C^{1/3}$ to turn the above inequality into

$$\Psi(t) \le \frac{9C^{1/3}}{4}\theta^{2/3}t^{2/3} + \frac{C_1 t^{1/3}}{\theta^{2/3}} \exp\Bigl(\frac{-\theta^{4/3}t^{1/3}}{C_1}\Bigr) + 2Ce^{-t/C}.$$

The second term on the right above forces us to restrict t further. We can fix a constant c_0 large enough so that, for a new constant C,

$$\Psi(t) \le C\theta^{2/3}t^{2/3} \quad \text{provided } t \ge c_0\theta^{-4}. \tag{5.37}$$

Restrict to t that satisfy this requirement and substitute this bound on $\Psi(t)$ into (5.36). Then upon using $u \ge B\theta^{2/3}t^{2/3}$ and redefining C once more, we have for $B\theta^{2/3}t^{2/3} \le u \le 20t/3$:

$$\mathbf{P}^\rho\{\, |Q(t) - V^\rho t| \ge u\} \le C\frac{\theta^2 t^2}{u^3} + 2e^{-u^2/Ct}. \tag{5.38}$$

Now take $1 < m < 3$ and use (5.38) together with the second case of (5.36):

$$\begin{aligned}
\mathbf{E}^\rho|Q(t) - V^\rho t|^m &= m\int_0^\infty \mathbf{P}^\rho\{\, |Q(t) - V^\rho t| \ge u\}u^{m-1}\,du \\
&\le B^m\theta^{2m/3}t^{2m/3} \;+\; Cm\theta^2t^2\int_{B\theta^{2/3}t^{2/3}}^\infty u^{m-4}\,du \\
&\quad+\; 2m\int_{B\theta^{2/3}t^{2/3}}^\infty e^{-u^2/Ct}u^{m-1}\,du \;+\; 2m\int_{20t/3}^\infty e^{-u/C}u^{m-1}\,du.
\end{aligned}$$

Performing and approximating the integrals gives

$$\mathbf{E}^\rho|Q(t) - V^\rho t|^m \le \frac{C}{3-m}\theta^{2m/3}t^{2m/3}$$

provided $t \ge c_0\theta^{-4}$ for a large enough constant c_0. □

5.6. Proof of the lower bound for second class particle moments

By Jensen's inequality it suffices to prove the lower bound for $m = 1$. Let C_{UB} denote the constant in the upper bound statement that we just proved. We can also assume $c_0 \ge 1$. Fix a constant $b > 0$ and set

$$a_1 = 2C_{UB} + 1 \quad \text{and} \quad a_2 = 8 + \sqrt{32b} + 8\sqrt{C_{UB}}.$$

Increase b if necessary so that

$$b^2 - 2a_2 \ge 1. \tag{5.39}$$

Fix a density $\rho \in (0,1)$ and define an auxiliary density $\lambda = \rho - bt^{-1/3}\theta^{-1/3}$. Define positive integers

$$u = \lfloor a_1 t^{2/3}\theta^{2/3}\rfloor \quad \text{and} \quad n = \lfloor V^\lambda t\rfloor - \lfloor V^\rho t\rfloor + u. \tag{5.40}$$

By taking c_0 large enough in the statement of Theorem 5.2 we can ensure that $\lambda \in (\rho/2, \rho)$ and $u \in \mathbb{N}$.

Construct a basic coupling of three processes $\eta \le \eta^+ \le \zeta$ with the following initial state:

(a) Initially η has i.i.d. Bernoulli(λ) occupations $\{\eta_i(0) : i \ne -n\}$ and $\eta_{-n}(0) = 0$.

(b) Initially $\eta^+(0) = \eta(0) + \delta_{-n}$. $Q^{(-n)}(t)$ is the location of the unique discrepancy between $\eta(t)$ and $\eta^+(t)$.

(c) Initially ζ has independent occupation variables, coupled with $\eta(0)$ as follows:

(c.1) $\zeta_i(0) = \eta_i(0)$ for $-n < i \le 0$.

(c.2) $\zeta_{-n}(0) = 1$.

(c.3) For $i < -n$ and $i > 0$ variables $\zeta_i(0)$ are i.i.d. Bernoulli(ρ) and $\zeta_i(0) \ge \eta_i(0)$.

Thus the initial density of ζ is piecewise constant: on the segment $\{-n+1,\dots,0\}$ $\zeta(0)$ is i.i.d. with density λ, at site $-n$ $\zeta(0)$ has density 1, and elsewhere on $\mathbb{Z}$ $\zeta(0)$ is i.i.d. with density ρ. The reason for the gap in the $\zeta-\eta$ second class particles across $(-n,0]$ is to get an upper bound on the second-class particle current that is not too large for subsequent arguments ((5.44) below).

Label the $\zeta-\eta$ second class particles as $\{Y_m(t) : m \in \mathbb{Z}\}$ so that initially

$$\cdots < Y_{-1}(0) < Y_0(0) = -n = Q^{(-n)}(0) < 0 < Y_1(0) < Y_2(0) < \cdots$$

Let again $m_Q(t)$ be the label such that $Q^{(-n)}(t) = Y_{m_Q(t)}(t)$. Initially $m_Q(0) = 0$. The inclusion $Q^{(-n)}(t) \in \{Y_m(t)\}$ persists for all time because the basic coupling preserves the ordering $\zeta(t) \geq \eta^+(t)$. Through the basic coupling m_Q jumps to the left with rate q and to the right with rate p, but only when there is a Y-particle adjacent to Y_{m_Q}. As in the proof of Lemma 5.8 we can apply Lemma 5.6 to prove this statement:

$$\mathbf{P}\{m_Q(t) \leq -k\} \leq e^{-2\theta k}. \tag{5.41}$$

By the upper bound already proved and by the choice of a_1,

$$\begin{aligned}\mathbf{P}\{Q^{(-n)}(t) \geq \lfloor V^\rho t\rfloor\} &= \mathbf{P}\{Q^{(-n)}(t) \geq -n + \lfloor V^\lambda t\rfloor + u\}\\ &\leq u^{-1}\mathbf{E}|Q^{(-n)}(t) - n - \lfloor V^\lambda t\rfloor| \leq \frac{C_{UB}t^{2/3}\theta^{2/3}}{\lfloor a_1 t^{2/3}\theta^{2/3}\rfloor}\\ &\leq \tfrac{1}{2}.\end{aligned} \tag{5.42}$$

For the complementary event we get a lower bound:

$$\begin{aligned}\tfrac{1}{2} &\leq \mathbf{P}\{Q^{(-n)}(t) \leq \lfloor V^\rho t\rfloor\}\\ &\leq \mathbf{P}\{m(t) \leq -k\} + \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t) \leq k\}.\end{aligned} \tag{5.43}$$

The reasoning behind the second inequality above is this. If $Q^{(-n)}(t) \leq \lfloor V^\rho t\rfloor$ and $m_Q(t) > -k$ then $Y_{-k}(t) \leq \lfloor V^\rho t\rfloor$. This implies a bound on the second class particle current:

$$J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t) = J^{\zeta-\eta}_{\lfloor V^\rho t\rfloor}(t) \leq k. \tag{5.44}$$

Put $k = \lfloor a_2 t^{1/3}\theta^{1/3}\rfloor - 2$. Then by $t \geq \theta^{-4}$ and the definition of a_2,

$$\mathbf{P}\{m_Q(t) \leq -k\} \leq e^{-2} < 1/4. \tag{5.45}$$

Combine (5.43) and (5.45) and split the probability:

$$\begin{aligned}\tfrac{1}{4} &\leq \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t) \leq a_2 t^{1/3}\theta^{1/3} - 2\}\\ &\leq \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) \leq 2a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda - \lambda^2)\}\\ &\quad + \mathbf{P}\{J^\eta_{\lfloor V^\rho t\rfloor}(t) \geq a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda - \lambda^2) + 2\}.\end{aligned} \tag{5.46}$$

Consider next line (5.46). The η-process can be coupled with a stationary P^λ-process with at most one discrepancy. The mean current in the stationary process is

$$\begin{aligned}E^\lambda[J_{\lfloor V^\rho t\rfloor}(t)] &= tH(\lambda) - \lambda\lfloor V^\rho t\rfloor\\ &\leq tH(\lambda) - \lambda V^\rho t + 1 = t\theta(2\rho\lambda - \lambda^2) + 1.\end{aligned}$$

Hence

$$\begin{aligned}
\text{line (5.46)} &\le P^{\lambda}\{J_{\lfloor V^\rho t\rfloor}(t) \ge a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda-\lambda^2)+1\}\\
&\le P^{\lambda}\big\{\overline{J}_{\lfloor V^\rho t\rfloor}(t) \ge a_2 t^{1/3}\theta^{1/3}\big\} \le a_2^{-2}t^{-2/3}\theta^{-2/3}\,\mathrm{Var}^{\lambda}\big[J_{\lfloor V^\rho t\rfloor}(t)\big]\\
&\le \frac{\mathbf{E}^{\lambda}|Q(t)-\lfloor V^\rho t\rfloor|}{a_2^2 t^{2/3}\theta^{2/3}} \;\le\; \frac{\mathbf{E}^{\lambda}|Q(t)-V^{\lambda}t|}{a_2^2 t^{2/3}\theta^{2/3}} + \frac{2b}{a_2^2} + \frac{1}{a_2^2 t^{2/3}\theta^{2/3}}\\
&\le C_{UB}a_2^{-2} + \tfrac{1}{16} + \tfrac{1}{64} \le \tfrac{1}{8}. \qquad (5.47)
\end{aligned}$$

After Chebyshev above we applied the basic identity (5.8) for which we introduced a second class particle $Q(t)$ in a density-λ system under the measure $\mathbf{P}^{\lambda}$. Then we replaced $\lfloor V^\rho t\rfloor$ with $V^{\lambda}t$ and applied the upper bound and properties of a_2.

Put this last bound back into line (5.46) to be left with

$$\tfrac{1}{8} \le \mathbf{P}\{J^{\zeta}_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda-\lambda^2)\}. \qquad (5.48)$$

Next we replace the ζ-process with a stationary density-ρ process by inserting the Radon-Nikodym factor. Let γ denote the distribution of the initial $\zeta(0)$ configuration described by (c1)–(c3) in the beginning of this section. As before ν^ρ is the density-ρ i.i.d. Bernoulli measure. The Radon-Nikodym derivative is

$$f(\omega) = \frac{d\gamma}{d\nu^\rho}(\omega) = \frac{1}{\rho}\mathbf{1}\{\omega_{-n}=1\}\cdot \prod_{i=-n+1}^{0}\Big(\frac{\lambda}{\rho}\mathbf{1}\{\omega_i=1\} + \frac{1-\lambda}{1-\rho}\mathbf{1}\{\omega_i=0\}\Big).$$

Bound its second moment:

$$E^{\rho}(f^2) = \frac{1}{\rho}\Big(1+\frac{(\rho-\lambda)^2}{\rho(1-\rho)}\Big)^n \le \rho^{-1}e^{n(\rho-\lambda)^2/\rho(1-\rho)} \le c_2(\rho) \qquad (5.49)$$

where condition $t \ge c_0\theta^{-4}$ implies a bound $c_2(\rho) < \infty$ independent of t and θ.

Let $\mathcal{A}$ denote the exclusion process event

$$\mathcal{A} = \{J_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda-\lambda^2)\}.$$

Then from (5.48)

$$\begin{aligned}
\tfrac{1}{8} \le \mathbf{P}\{\zeta\in\mathcal{A}\} &= \int P^{\omega}(\mathcal{A})\,\gamma(d\omega) = \int P^{\omega}(\mathcal{A})f(\omega)\,\nu^\rho(d\omega)\\
&\le \big(P^{\rho}(\mathcal{A})\big)^{1/2}\big(E^{\rho}(f^2)\big)^{1/2} \le c_2(\rho)^{1/2}\big(P^{\rho}(\mathcal{A})\big)^{1/2}. \qquad (5.50)
\end{aligned}$$

Note the stationary mean

$$E^{\rho}\big[J_{\lfloor V^\rho t\rfloor}(t)\big] = tH(\rho) - \rho\lfloor V^\rho t\rfloor = t\theta\rho^2 + \rho V^\rho t - \rho\lfloor V^\rho t\rfloor \ge t\theta\rho^2.$$

Continue from line (5.50), recalling (5.39):

$$\begin{aligned}
(64c_2(\rho))^{-1} &\le P^{\rho}(\mathcal{A}) = P^{\rho}\{J_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + t\theta(2\rho\lambda-\lambda^2)\}\\
&\le P^{\rho}\{\,\overline{J}_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} - t\theta(\rho-\lambda)^2\}\\
&= P^{\rho}\{\,\overline{J}_{\lfloor V^\rho t\rfloor}(t) \le -(b^2-2a_2)t^{1/3}\theta^{1/3}\} \le P^{\rho}\{\,\overline{J}_{\lfloor V^\rho t\rfloor}(t) \le -t^{1/3}\theta^{1/3}\}\\
&\le t^{-2/3}\theta^{-2/3}\,\mathrm{Var}^{\rho}\big[J_{\lfloor V^\rho t\rfloor}(t)\big] \;\le\; t^{-2/3}\theta^{-2/3}\mathbf{E}^{\rho}|Q(t)-V^\rho t|.
\end{aligned}$$

This completes the proof of the lower bound. We finish with some observations about the need for the two key assumptions, asymmetry and $H''(\rho)\neq 0$.

For symmetric SEP $\theta = 0$ and consequently the Chebyshev step above cannot be taken.

To observe where $H''(\rho) < 0$ came in we need to backtrack a little. At stage (5.48) we have the inequality (ignoring now small errors due to integer parts etc.)

$$\tfrac{1}{8} \le \mathbf{P}\big\{J^{\zeta}_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + E^{\lambda}(J_{\lfloor V^\rho t\rfloor}(t))\big\}.$$

The Radon-Nikodym and Schwarz trick turned this into an inequality for a stationary process:

$$\begin{aligned} 0 < c &\le P^{\rho}\big\{J_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + E^{\lambda}(J_{\lfloor V^\rho t\rfloor}(t))\big\} \\ &= P^{\rho}\big\{\overline{J}_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3}\theta^{1/3} + E^{\lambda}(J_{\lfloor V^\rho t\rfloor}(t)) - E^{\rho}(J_{\lfloor V^\rho t\rfloor}(t))\big\}. \end{aligned} \tag{5.51}$$

Compute the means on the right-hand side inside the probability, remembering that $V^\rho = H'(\rho)$ and Taylor expanding $H(\lambda)$:

$$\begin{aligned} E^{\lambda}(J_{\lfloor V^\rho t\rfloor}(t)) - E^{\rho}(J_{\lfloor V^\rho t\rfloor}(t)) &= t\big[H(\lambda) - \lambda H'(\rho) - H(\rho) + \rho H'(\rho)\big] \\ &= t\big[\tfrac{1}{2}H''(\rho)(\lambda-\rho)^2 + O\big(\theta|\lambda-\rho|^3\big)\big] \\ &= -a_3 b^2 t^{1/3}\theta^{1/3} + O(1) \end{aligned}$$

with $\frac{1}{2}H''(\rho) = -a_3\theta < 0$ in the last step. Thus the constants can be adjusted so that the probability in (5.51) is a deviation. Chebyshev can be applied to conclude that the current variance is of order $t^{2/3}\theta^{2/3}$. But if $H''(\rho) = 0$ there is no deviation to take advantage of.

Problems

Exercise 5.1. When infinite particle systems such as ASEP are constructed with Poisson clocks, there is an issue of well-definedness that needs to be resolved. Namely, if we ask whether site x is occupied at time t, we need to look backwards in time at all the possible sites from which a particle could have moved to x by time t. This might involve an infinite regression: perhaps there is a sequence of times $t > t_1 > t_2 > t_3 > \cdots > 0$ such that Poisson clock $N^{x-k,x-k+1}$ signaled a jump attempt at time t_k. Such a sequence of jumps could in principle bring a particle to x "from infinity." How do you argue that this is not a problem?

How would you do it if you needed to construct ASEP on a higher dimensional lattice $\mathbb{Z}^d$ for $d \ge 2$?

Also, when we described the rules of ASEP, we made no provision for the possibility of simultaneous jumps. What simple fact enables us to ignore this issue?

Exercise 5.2. It is probably familiar that the infinitesimal evolution of a continuous-time Markov chain is expressed in terms of a generator matrix. The corresponding notion for interacting particle systems is the (infinitesimal) *generator* as an operator on functions on the state space of the system. The generator of the TASEP ($p = 1 = 1 - q$) is

$$L\varphi(\eta) = \sum_{i\in\mathbb{Z}} \eta_i(1-\eta_{i+1})[\varphi(\eta^{i,i+1}) - \varphi(\eta)] \tag{5.52}$$

that acts on cylinder functions φ on the state space $\{0,1\}^{\mathbb{Z}}$ and $\eta^{i,i+1} = \eta - \delta_i + \delta_{i+1}$ is the configuration that results from moving one particle from site i to $i+1$. In general, the generator of a process is defined as the derivative of the semigroup:

$$L\varphi(\eta) = \lim_{t\searrow 0} \frac{E^{\eta}[f(\eta(t))] - f(\eta)}{t}. \tag{5.53}$$

Above E^η denotes expectation under P^η, the distribution of the process when the initial state is η.

Derive (5.53) from the graphical construction in terms of Poisson clocks, at least in some hand-waiving manner.

EXERCISE 5.3. As in continuous-time Markov chains, invariance of a distribution can be checked by a generator computation. For TASEP, it is enough to check that

$$\int L\varphi \, d\mu = 0 \tag{5.54}$$

for cylinder functions φ to conclude that μ is invariant. Use this to check that Bernoulli measures ν^ρ are invariant for TASEP.

What more do you need to say to conclude the same for ASEP?

EXERCISE 5.4. Let $\omega(0) \geq \eta(0)$ be two initial configuration for ASEP. Run the pair process $(\eta(t), \omega(t))$ in basic coupling, that is, letting both processes obey the same Poisson clocks $\{N^{i\to i+1}, N^{i\to i-1}\}$. Show that the ordering $\omega(t) \geq \eta(t)$ holds for all time.

EXERCISE 5.5. Show that (5.2) defines a martingale for TASEP (again, it may be best to include some hand-waiving instead of striving for full rigor). Then derive (5.1) and (5.3).

EXERCISE 5.6. It follows from the variance bound (5.11) that for $v \neq V^\rho$ a Gaussian limit in the central limit scale holds:

$$\frac{J_{[tv]}(t) - E^\rho(J_{[tv]}(t))}{t^{1/2}} \xrightarrow{\mathcal{D}} \chi \tag{5.55}$$

for some centered normal random variable χ. *Hint:* Let us suppose $v > V^\rho$. Let J^* be the current across the straight-line space-time path from $((v - V^\rho)t, 0)$ to (vt, t). This current has variance of order $t^{2/3}$. Notice that

$$J^* = J_{[tv]}(t) + \sum_{i=1}^{(v-V^\rho)t} \eta_i(0).$$

EXERCISE 5.7. Prove (5.18).

Further comments and references

The proofs of this chapter are based on [**BS09a**]. This article gives simpler proofs for the results in [**BS07**] and [**BS09b**]. Precursors of these variance bounds were first proved for last-passage models that correspond to totally asymmetric versions of particle systems: in [**CG06**] for the Hammersley process and in [**BCS06**] for the corner growth model associated with TASEP.

The Tracy-Widom type limit distribution for TASEP current was first proved for the step initial condition in [**Joh00**], then for the stationary case (Theorem 5.3) in [**FS06**]. The larger picture of TASEP fluctuations from various initial conditions is presented in [**BAC09**]. For ASEP the limit distribution has been proved for the step initial condition in [**TW**].

Another line of work has produced comparison theorems that allow one to conclude that Laplace transforms of $t^{-1}\mathbf{Var}^\rho[Q(t)]$ for different asymmetric exclusion processes are within constant multiples of each other. In this sense, for the order of this Laplace transform there is universality for all finite range asymmetric exclusion processes. These results come from resolvent techniques [**Set03, QV07, QV08**].

The central limit theorem for the current in directions other than the characteristic V^ρ was proved first by Ferrari and Fontes [**FF94**]. This was generalized to other particle systems such as certain zero-range and bricklayer processes by Balázs [**Bal03**].

Consider symmetric simple exclusion, namely the case $p = q = 1/2$. Then $V^\rho = 0$. Equation (5.8) together with the observation that the second class particle is a simple symmetric random walk tell us that $\mathrm{Var}^\rho[J_0(t)]$ is of order $t^{1/2}$, exactly as for independent particles in Chapter 2. And indeed the current process does converge to fractional Brownian motion (see [**PS08**] and its references).

CHAPTER 6

Zero range process

6.1. Model and results

From the perspective of universality it would be highly desirable to extend the results of Chapter 5 beyond exclusion processes. Throughout the 40-year history of the subject of interacting particle systems, the *zero range process* has been a much-studied relative of the exclusion process. In this section we indicate how the bounds for second class particles and current variance are proved for a class of totally asymmetric zero range processes (TAZRP) with concave jump rate functions.

Definition and graphical construction. In contrast with the exclusion process, the zero range process does not restrict the number of particles allowed at a site. The state of the process at time t is $\eta(t) = (\eta_i(t))_{i\in\mathbb{Z}} \in \mathbb{Z}_+^{\mathbb{Z}}$ where $\eta_i(t) \in \mathbb{Z}_+$ denotes the number of particles present at site i at time t. We consider the case where particles take only nearest-neighbor jumps to the right.

Each zero range process is characterized by a jump rate function $g : \mathbb{Z}_+ \to \mathbb{R}_+$. It automatically has the value $g(0) = 0$. The meaning of g for the process is that when the current state is η, $g(\eta_i)$ is the rate at which one particle is moved from site i to $i + 1$. You can interpret this as saying that each of the particles at site i jumps independently with rate $\eta_i^{-1}g(\eta_i)$ or that some particular one moves next (say, the bottom one is moved to the top of the next pile) at rate $g(\eta_i)$. It is immaterial for we do not label our particles. (Except again we will label certain second class particles as we did for ASEP, but we will come to that later.) These jump events take place independently at all sites, exactly as for ASEP.

We shall assume that

$$g \text{ is nondecreasing and also that } 0 < g(k) \le 1 \text{ for } k > 0. \tag{6.1}$$

With a bounded g we can perform the following concrete construction of the process $\eta(t)$ in terms of independent rate 1 Poisson processes $\{N_i\}$ and i.i.d. Uniform$(0,1)$ variables $\{U_{i,k}\}$. Attach clock N_i to site i, and give each jump time of N_i its own $U_{i,k}$. Now if t is a jump time for N_i with its uniform $U_{i,k}$, then move one particle from i to $i+1$ if $U_{i,k} < g(\eta_i(t-))$, otherwise not. Repeat this step at all sites and all jump times. The result is the desired one: independently at each site i, a jump from i to $i+1$ occurs in the next small time interval $(t, t+dt)$ with probability $g(\eta_i(t))dt + O(dt^2)$.

The dynamics of Markov processes are conveniently expressed in terms of operators called *generators*. The generator of this TAZRP is

$$L\varphi(\eta) = \sum_{i\in\mathbb{Z}} g(\eta_i)[\varphi(\eta^{i,i+1}) - \varphi(\eta)]$$

that acts on bounded cylinder functions φ on $\mathbb{Z}_+^{\mathbb{Z}}$ and $\eta^{i,i+1} = \eta - \delta_i + \delta_{i+1}$. We will not use the generator in the text. (In the exercises it is used to check the invariance of certain distributions on the state space, as for ASEP.)

Invariant distributions. Part of the reason for the popularity of ZRP is that, just like ASEP, it has i.i.d. invariant distributions. We denote these $\{\nu^\rho\}_{0\le\rho<\infty}$ indexed by density $\rho = E^\rho(\eta_i)$.

Here is the definition of these measures. Let θ denote a real parameter, and on $\mathbb{Z}_+$ define a probability distribution

$$\lambda^\theta(k) = \frac{1}{Z_\theta} \cdot \frac{e^{\theta k}}{g(k)!},$$

defined for θ such that

$$Z_\theta = \sum_k \frac{e^{\theta k}}{g(k)!} < \infty.$$

Here $g(0)! = 1$ and $g(k)! = g(1) \cdots g(k)$ for $k > 0$. Define the mean density function $\rho(\theta) = \sum_k k\lambda^\theta(k)$. It is smooth and strictly increasing on the open interval where $Z_\theta < \infty$. Let its inverse function be $\theta(\rho)$ and then reparametrize the distributions in terms of density:

$$\nu_0^\rho(k) = \lambda^{\theta(\rho)}(k) = \frac{1}{Z_{\theta(\rho)}} \cdot \frac{e^{\theta(\rho)k}}{g(k)!}.$$

Finally, the actual invariant measures for ZRP are the product measures on the state space $\mathbb{Z}_+^{\mathbb{Z}}$:

$$\nu^\rho(d\eta) = \bigotimes_{i \in \mathbb{Z}} \nu_0^\rho(d\eta_i). \tag{6.2}$$

We write P^ρ for probabilities and E^ρ for expectations for the stationary process whose marginal $\eta(t)$ has distribution ν^ρ.

Basic coupling and second class particles. *Basic coupling* works exactly as it did for exclusion processes: two (or more) zero range processes obey a common set of Poisson clocks $\{N_i\}$ and uniform variables $\{U_{i,k}\}$.

We write a boldface $\mathbf{P}$ for the probability measure when more than one process are coupled together. In particular, $\mathbf{P}^\rho$ represents the situation where the initial occupation variables $\eta_i(0) = \eta_i^+(0)$ are i.i.d. mean-ρ Bernoulli for $i \neq 0$, and the second class particle Q starts at $Q(0) = 0$.

More generally, if two processes η and ω are in basic coupling and $\omega(0) \geq \eta(0)$ (by which we mean coordinatewise ordering $\omega_i(0) \geq \eta_i(0)$ for all i) then the ordering $\omega(t) \geq \eta(t)$ holds for all $0 \leq t < \infty$. The effect of the basic coupling is to give priority to the η particles over the $\omega - \eta$ particles. Consequently we can think of the ω-process as consisting of first class particles (the η particles) and second class particles (the $\omega - \eta$ particles).

Current. The current is defined as for ASEP: for $x \in \mathbb{Z}$ and $t > 0$, $J_x(t)$ is the net left-to-right particle current across the straight-line space-time path from $(1/2, 0)$ to $(x + 1/2, t)$.

The flux function is again

$$H(\rho) = E^\rho[\text{rate of particle flow across a fixed edge}] = E^\rho[g(\eta_i)].$$

Expectations of currents can be computed as for ASEP:

$$E^\rho[J_x(t)] = tH(\rho) - x\rho, \qquad x \in \mathbb{Z}, t \geq 0. \tag{6.3}$$

The *characteristic speed* at density ρ is defined the same way as before:

$$V^\rho = H'(\rho). \tag{6.4}$$

As for ASEP, the first task is to establish the identities that connect current variance and the second class particle. These identities for ZRP develop the same way as for ASEP, except that a new initial distribution for the coupled process appears. Define a probability distribution $\hat{\nu}_0^\rho$ on $\mathbb{Z}_+$ by

$$\hat{\nu}_0^\rho(k) = \frac{1}{\mathrm{Var}^\rho(\eta_0)} \sum_{m=k+1}^{\infty} (m - \rho)\nu_0^\rho(m), \qquad k \in \mathbb{Z}_+.$$

Define a product distribution $\hat\nu^\rho$ on the state space $\mathbb{Z}_+^{\mathbb{Z}}$ that obeys the marginals ν_0^ρ of the stationary distribution at all sites except at the origin where the distribution is $\hat\nu_0^\rho$:

$$\hat\nu^\rho(d\eta) = \Big(\bigotimes_{i\neq 0} \nu_0^\rho(d\eta_i)\Big) \otimes \hat\nu_0^\rho(d\eta_0).$$

Let $\widehat{\mathbf{P}}^\rho$ be the probability distribution of a pair (η, η^+) that satisfies $\eta^+(t) = \eta(t) + \delta_{Q(t)}$ (so there is one discrepancy), obeys basic coupling, and whose initial distribution is such that $\eta(0) \sim \hat\nu^\rho$ and $\eta^+(0) = \eta(0) + \delta_0$ (in other words, $Q(0) = 0$).

Theorem 6.1. *For any density* $0 < \rho < \infty$, $z \in \mathbb{Z}$ *and* $t > 0$ *we have these formulas.*

$$\mathrm{Var}^\rho[J_z(t)] = \sum_{j\in\mathbb{Z}} |j - z| \,\mathrm{Cov}^\rho[\eta_j(t), \eta_0(0)], \tag{6.5}$$

$$\mathrm{Cov}^\rho[\eta_j(t),\, \eta_0(0)] = \mathrm{Var}^\rho(\eta_0)\widehat{\mathbf{P}}^\rho\{Q(t) = j\}, \tag{6.6}$$

and

$$\widehat{\mathbf{E}}^\rho[Q(t)] = V^\rho t. \tag{6.7}$$

Equation (6.5) is proved the same way as for ASEP. Equation (6.6) and the definition of $\hat\nu_0^\rho$ come from a short calculation which we show below in Section 6.2. We omit the proof of (6.7). Formulas (6.5) and (6.6) combine to give the key equation that links current variance with the second class particle:

$$\mathrm{Var}^\rho[J_z(t)] = \mathrm{Var}^\rho(\eta_0)\widehat{\mathbf{E}}^\rho|Q(t) - z|. \tag{6.8}$$

Next we state the main result which again consists of upper and lower moment bounds for a second class particle, this time under the measure $\widehat{\mathbf{P}}^\rho$. We need a significant restriction on the concavity of the jump rate g:

$$\exists\ 0 < r < 1 \text{ such that } g(k + 1) - g(k) \le r(g(k) - g(k - 1)). \tag{6.9}$$

A class of examples satisfying this hypotheses is given by $g(k) = 1 - \exp(-ak^b)$ with $a > 0$, $b \ge 1$. $g(k)$ can also be constant from some k_0 onwards.

Theorem 6.2. *Fix a density* $0 < \rho < \infty$ *and consider a pair of coupled ZRP's under the measure* $\widehat{\mathbf{P}}^\rho$. *Assume the jump rate function* g *satisfies* (6.1) *and* (6.9). *Then for* $1 \le m < 3$, *large enough* $t \in \mathbb{R}_+$ *and a constant* C,

$$\frac{1}{C} t^{2m/3} \le \widehat{\mathbf{E}}^\rho\big[\, |Q(t) - V^\rho t|^m \,\big] \le \frac{C}{3 - m} t^{2m/3}. \tag{6.10}$$

The constants C and how large t needs to be may depend on the density ρ. Combining (6.5) and (6.10) gives the bounds for the variance of the current seen by an observer traveling at the characteristic speed V^ρ: for large enough t,

$$C^{-1} t^{2/3} \le \mathrm{Var}^\rho[J_{\lfloor V^\rho t\rfloor}(t)] \le Ct^{2/3}.$$

For the remainder of this chapter we discuss parts of the proof. Once we have the fundamental identities that tie together moments of the second class particle and the variance of the current, the proofs for the upper and lower bounds given for ASEP in Sections 5.5 and 5.6 can be adjusted to work for TAZRP. We will not repeat those derivations. Instead, we focus on the key ingredients that made the argument work for ASEP, and discuss how to provide these ingredients for TAZRP. There are two key points that we need in order to repeat the argument for TAZRP:

(1) We need a construction that includes a given second class particle as a labeled member of a density of second class particles, and then we need a tail bound for the label as the one given for ASEP in Lemma 5.8 and (5.41).
(2) We need a coupling that keeps the second class particle of a system with higher density behind the second class particle of a system with lower density. For ASEP this was Theorem 5.4, which was used to obtain (5.33) for the ASEP upper bound.

We turn to these points in Section 6.3 below, after developing the variance formula.

6.2. Variance identity

Define $F(-1) = 0$ and

$$F(k) = \sum_{m=k+1}^{\infty} (m-\rho)\frac{\nu_0^\rho(m)}{\nu_0^\rho(k)} = \frac{\mathrm{Var}^\rho(\eta_0)}{\nu_0^\rho(k)}\hat{\nu}_0^\rho(k), \quad k \ge 0.$$

Interpret below $\nu_0^\rho(-1)$ as 0.

$$\begin{aligned}\mathrm{Cov}^\rho[\eta_i(t),\eta_0(0)] &= E^\rho[\eta_i(t)(\eta_0(0)-\rho)] = \sum_{k\ge 0} E[\eta_i(t)|\eta_0(0)=k](k-\rho)\nu_0^\rho(k)\\ &= \sum_{k\ge 0} E[\eta_i(t)|\eta_0(0)=k]\big(F(k-1)\nu_0^\rho(k-1) - F(k)\nu_0^\rho(k)\big)\\ &= \sum_{k\ge 0} E[\eta_i(t)|\eta_0(0)=k+1]F(k)\nu_0^\rho(k) - \sum_{k\ge 0} E[\eta_i(t)|\eta_0(0)=k]F(k)\nu_0^\rho(k).\end{aligned}$$

Construct a coupling of $\eta^+(t) = \eta(t) + \delta_{Q(t)}$ with the discrepancy initially at the origin $Q(0) = 0$, and so that $\eta(0)$ has ν^ρ-distribution. Then, due to the product form of the initial distribution,

$$\mathbf{E}[\eta_i(t)|\eta_0(0)=k+1] = \mathbf{E}[\eta_i^+(t)|\eta_0^+(0)=k+1] = \mathbf{E}[\eta_i^+(t)|\eta_0(0)=k].$$

Then continuing from above,

$$\begin{aligned}\mathrm{Cov}^\rho[\eta_i(t),\eta_0(0)] &= \sum_{k\ge 0}\mathbf{E}[\eta_i^+(t)|\eta_0(0)=k]F(k)\nu_0^\rho(k) - \sum_{k\ge 0}\mathbf{E}[\eta_i(t)|\eta_0(0)=k]F(k)\nu_0^\rho(k)\\ &= \sum_{k\ge 0}\mathbf{E}[\eta_i^+(t)-\eta_i(t)|\eta_0(0)=k]F(k)\nu_0^\rho(k)\\ &= \mathbf{E}\big[(\eta_i^+(t)-\eta_i(t))F(\eta_0(0))\big]\\ &= \mathbf{E}\big[\mathbf{1}\{Q(t)=i\}F(\eta_0(0))\big]\\ &= \sum_{k\ge 0}\mathbf{E}[\mathbf{1}\{Q(t)=i\}|\eta_0(0)=k]F(k)\nu_0^\rho(k)\\ &= \mathrm{Var}^\rho(\eta_0)\sum_{k\ge 0}\mathbf{E}[\mathbf{1}\{Q(t)=i\}|\eta_0(0)=k]\hat{\nu}_0^\rho(k)\\ &= \mathrm{Var}^\rho(\eta_0)\widehat{\mathbf{P}}^\rho\{Q(t)=i\}.\end{aligned}$$

This proves (6.6).

6.3. Coupling for the zero range process

Next we describe a coupling of two processes with labeled discrepancies (second class particles) between them, and then two randomly evolving labels that achieve simultaneously both goals (1) and (2) mentioned above. This construction works for any TAZRP with concave jump rate function g. Getting tail bounds for the label processes is the serious bottleneck of this proof, and that is where we need the restrictive assumption (6.9).

Let two processes $\eta \le \omega$ evolve in basic coupling. This pair (η, ω) together with the labeled and ordered $\omega - \eta$ second class particles $\cdots \le X_{-2}(t) \le X_{-1}(t) \le X_0(t) \le X_1(t) \le X_2(t) \le \cdots$ form a "background" process on which we define two label processes $y(t)$ and $z(t)$. The $\omega - \eta$ second class particles are kept in order by requiring that, whenever a second class particle jumps to the right, the X-particle with highest label is moved.

The label processes will satisfy $y(t) \le z(t)$, we will be able to bound $y(t)$ stochastically from above, $z(t)$ stochastically from below, and the following two pairs of processes will individually be in basic coupling:

$$(\omega^-, \omega) = (\omega - \delta_{X_y}, \omega) \quad \text{and} \quad (\eta, \eta^+) = (\eta, \eta + \delta_{X_z}). \tag{6.11}$$

The definition of the label processes is partly forced on us by the requirement that jumps of X_y and X_z must replicate the rates required by the basic coupling. Additionally, we devise the joint process (y, z) so that the order $y \le z$ is maintained.

The rule is that after each jump among (ω, η) that in any way affects the site where X_y resides, the value of y is refreshed randomly. Let a and b denote the minimal and maximal labels at the site i where X_y resides *after* the jump. If X_y resides at a site other than X_z, then y chooses a new value y' according to these probabilities:

$$y' = \begin{cases} a & \text{with probability } \dfrac{g(\omega_i - 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)} \\[2ex] b & \text{with probability } \dfrac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)}. \end{cases} \tag{6.12}$$

If $g(\omega_i) - g(\eta_i) = 0$ then $y' = a$.

Similarly, after a jump in the background process that affects the site where X_z resides, if X_z and X_y are not together, then z takes the new value z' as follows (with b again the maximal label at the site $i = X_z$ after the jump):

$$z' = \begin{cases} b - 1 & \text{with probability } \dfrac{g(\omega_i) - g(\eta_i + 1)}{g(\omega_i) - g(\eta_i)} \\[2ex] b & \text{with probability } \dfrac{g(\eta_i + 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)}. \end{cases}$$

If $g(\omega_i) - g(\eta_i) = 0$ then $z' = b$.

Finally, if after the jump $X_y = X_z = i$, then the labels are refreshed jointly as follows:

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{cases} \begin{pmatrix} a \\ b-1 \end{pmatrix} & \text{with probability } \dfrac{g(\omega_i) - g(\eta_i + 1)}{g(\omega_i) - g(\eta_i)} \\[2ex] \begin{pmatrix} a \\ b \end{pmatrix} & \text{with probability } \dfrac{g(\eta_i + 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)} - \dfrac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \\[2ex] \begin{pmatrix} b \\ b \end{pmatrix} & \text{with probability } \dfrac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)}. \end{cases}$$

If $g(\omega_i) - g(\eta_i) = 0$ then $(y', z') = (a, b)$.

The jump rules preserve $y \le z$. Note that marginally y' obeys probabilities (6.12), and similarly for z'. Concavity of g was used to define the middle case in the joint rule.

Let us observe why these rules give the pair $(\omega^-, \omega) = (\omega - \delta_{X_y}, \omega)$ the same rates this pair would have in basic coupling. The requirement is that a jump across edge $(i, i+1)$ occur for both processes at rate $g(\omega_i^-)$, and only for ω at rate $g(\omega_i) - g(\omega_i^-)$. This requires thinking through a few cases. Only the site where X_y resides needs attention since elsewhere (ω^-, ω) jump together according to ZRP rates.

(i) In the basic coupling of (η, ω), an $(i, i+1)$ jump occurs in η at rate $g(\eta_i)$. Then both ω and ω^- experience this jump.

(ii) An $\omega - \eta$ second class particle jumps at rate $g(\omega_i) - g(\eta_i)$. Prior to this jump y chose the top label with probability given by the second line of (6.12), hence the rate at which X_y jumps is

$$\big(g(\omega_i) - g(\eta_i)\big) \cdot \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} = g(\omega_i) - g(\omega_i - 1).$$

Thus at this rate ω experiences the jump but ω^- does not.

If prior to this jump y chose the bottom label, then both ω and ω^- experience this jump, and this happens with rate

$$\big(g(\omega_i) - g(\eta_i)\big) \cdot \frac{g(\omega_i - 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)} = g(\omega_i - 1) - g(\eta_i).$$

Adding up the rates we see that the rates of basic coupling have been realized. A similar argument is given for $(\eta, \eta^+) = (\eta, \eta + \delta_{X_z})$.

We come to the unique point in the proof where assumption (6.9) is used, namely the tail bounds for the labels.

LEMMA 6.3. *Let the labels start with* $y(0) = z(0) = 0$. *Under assumption* (6.9) *we have these bounds:* $\mathbf{P}\{y(t) \ge k\} \le r^k$ *and* $\mathbf{P}\{z(t) \le -k\} \le r^k$ *for all* $k \in \mathbb{Z}_+$ *and* $t \ge 0$.

PROOF. We do the proof for $y(t)$. The bounds are valid conditionally on the evolution (η, ω) of the background process. So assume this background evolution given. Then we think of $y(t)$ as an integer-valued Markov chain that is subject to jumps triggered by the background environment. Each jump happens at some site i with range of labels $\{a, \dots, b\}$ and occupation variables $\omega_i > \eta_i \ge 0$ that together satisfy

$$\omega_i - \eta_i = b - a + 1.$$

Given the current value y, the new value y' is obtained by the following rules, which of course are consistent with (6.12): if $g(\omega_i) - g(\eta_i) = 0$ then

$$y' = \begin{cases} y & \text{if } y < a \text{ or } y > b \\ a & \text{if } y \in \{a, \dots, b\} \end{cases} \tag{6.13}$$

while if $g(\omega_i) - g(\eta_i) > 0$ then

$$y' = \begin{cases} y & \text{if } y < a \text{ or } y > b \\ a & \text{with probability } \dfrac{g(\omega_i - 1) - g(\eta_i)}{g(\omega_i) - g(\eta_i)} \quad \text{if } y \in \{a, \dots, b\} \\ b & \text{with probability } \dfrac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \quad \text{if } y \in \{a, \dots, b\}. \end{cases} \tag{6.14}$$

There are infinitely many such jumps in any finite time interval, but all but finitely many leave y unchanged. To get around this difficulty, we can first freeze all the Poisson clocks outside space interval $[-M, M]$, prove the lemma there, and then let $M \nearrow \infty$. Since rates are bounded, in any given bounded block of space-time the finite-M process agrees with the infinite process for all large enough M.

To prove the lemma we show that every jump of type (6.13)–(6.14) preserves the geometric tail bound, regardless of the values a, b, ω_i, η_i. So suppose y is an integer-valued random variable such that

$$P(y \geq k) \leq r^k \quad \text{for} \quad k \geq 0,$$

and define y' via (6.13)–(6.14). We wish to show that $P(y' \geq k) \leq r^k$ for $k \geq 0$.

The case (6.13) is clear since there $y' \leq y$. Let us consider the case $g(\omega_i) - g(\eta_i) > 0$. Since the jump only redistributes the probability mass in $\{a, \dots, b\}$ to $\{a, b\}$, it suffices to check that

$$P(y' \geq b) \leq r^b \tag{6.15}$$

in the case $b \geq 0$. Using the jump rule (6.14),

$$P(y' \geq b) = P(y' = b) + P(y' \geq b+1) = \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} P(a \leq y \leq b) + P(y \geq b+1).$$

If $g(\omega_i) - g(\omega_i - 1) = 0$ the conclusion (6.15) follows from the assumption on y. So we assume $g(\omega_i) - g(\omega_i - 1) > 0$. Then by concavity all the g-increments between $\eta_i, \dots, \omega_i$ are positive. Next write

$$\begin{aligned} P(y' \geq b) &= \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \sum_{k=a}^{b} (1-r) r^k \\ &\quad + \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \bigl(P(a \leq y \leq b) - r^a + r^{b+1} \bigr) + P(y \geq b+1). \end{aligned} \tag{6.16}$$

For $a \leq k \leq b$

$$\begin{aligned} (1-r) r^k = (1-r) r^b \cdot \frac{1}{r^{b-k}} &\leq (1-r) r^b \prod_{\ell = \omega_i - b + k}^{\omega_i - 1} \frac{g(\ell) - g(\ell - 1)}{g(\ell+1) - g(\ell)} \\ &= (1-r) r^b \cdot \frac{g(\omega_i - b + k) - g(\omega_i - b + k - 1)}{g(\omega_i) - g(\omega_i - 1)}. \end{aligned}$$

Adding these up over $a \leq k \leq b$ gives

$$\begin{aligned} &\text{first term on the right in (6.16)} \\ &\leq \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} (1-r) r^b \sum_{k=a}^{b} \frac{g(\omega_i - b + k) - g(\omega_i - b + k - 1)}{g(\omega_i) - g(\omega_i - 1)} \\ &= (1-r) r^b \cdot \frac{g(\omega_i) - g(\omega_i - b + a - 1)}{g(\omega_i) - g(\eta_i)} = (1-r) r^b = r^b - r^{b+1}. \end{aligned}$$

Substitute this bound back up to (6.16) and use $P(y \geq k) \leq r^k$ twice:

$$\begin{aligned} P(y' \geq b) &\leq r^b + P(y \geq b+1) - r^{b+1} + \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \bigl(P(a \leq y \leq b) - r^a + r^{b+1} \bigr) \\ &\leq r^b + \frac{g(\omega_i) - g(\omega_i - 1)}{g(\omega_i) - g(\eta_i)} \bigl(P(y \geq a) - r^a \bigr) \\ &\leq r^b. \end{aligned}$$

Thus (6.15) has been checked and thereby the lemma has been proved for $y(t)$. $\square$

Problems

EXERCISE 6.1. Prove that the probability distributions ν^ρ defined in (6.2) are invariant. What you need to show is that $\int L\varphi \, d\nu^\rho = 0$ for cylinder functions φ. This comes readily by doing the right change of variable in the integral. To justify that this integral criterion is enough, you can first consider the process on a finite interval of sites with periodic boundary conditions. This is a countable state space Markov chain with bounded rates and by textbook theory this criterion is enough for invariance. Then justify that the process on the finite interval converges to the correct process the full lattice, by coupling these processes through the Poisson clocks. Alternately, look in [**And82**] for general theory of zero range processes.

EXERCISE 6.2. Think through the fact that the basic coupling preserves the ordering $\omega(t) \geq \eta(t)$.

References

The results for ZRP are proved in [**BKS08**].

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