

# Observables for $O(n)$ model on the honeycomb lattice. New game.

Alexander Glazman

University of Geneva; PDMI RAS, St. Petersburg

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joint work with Dmitry Chelkak and Stanislav Smirnov

# Outline

- Introduction (known results, motivation, definition of the model)
- New observable on edges (description, relations, specific weights giving them)
- New observable on midlines of hexagons
- Observables as the deformations of the metric ( $O(n)$  model on a rhombic tiling, critical weights, local non-planar deformations)
- Convergence result for the Ising model

## Observables as a tool

Observable is a function defined on a lattice. It contains information about particular statistical mechanical model and satisfies some local relations. If you have enough relations and can handle the observable on the boundary then you might hope to obtain convergence results.

2001, Conformal invariance of percolation on the triangular lattice (Smirnov)

2006, Conformal invariance of the Ising model on the square lattice (Smirnov)

2011, Universality of the 2D Ising model (Chelkak and Smirnov)

2011-..., other results in the Ising model (Hongler, Izyurov, Kytola)

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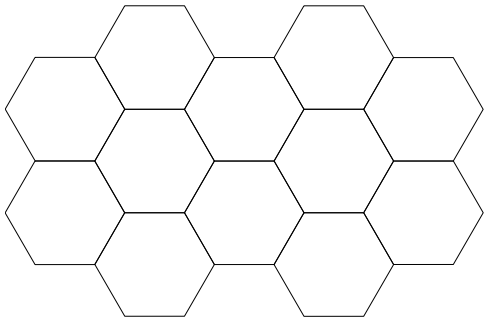
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# (Loop representation of) $O(n)$ model on the honeycomb lattice

Honeycomb lattice, finite part.

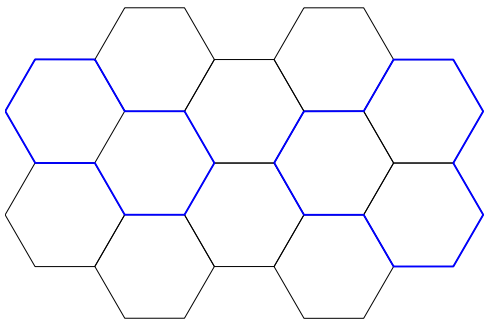


The configuration is a subgraph of this lattice where each vertex has degree 0 or 2. It can be divided into loops. The weight is calculated as follows:

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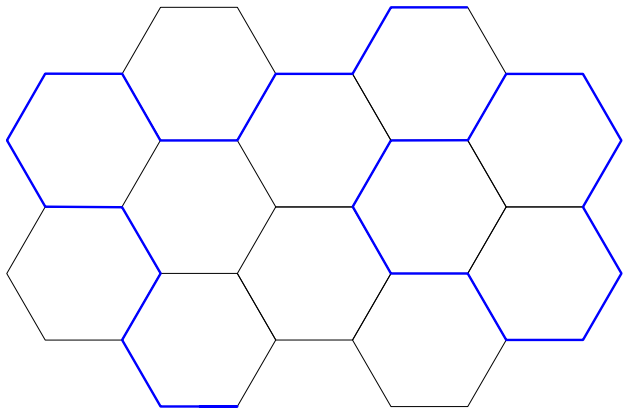


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## $O(n)$ model, boundary conditions

We can allow walks also. We just pick a pair of points on the boundary and say that they have degree 1 in our configuration.

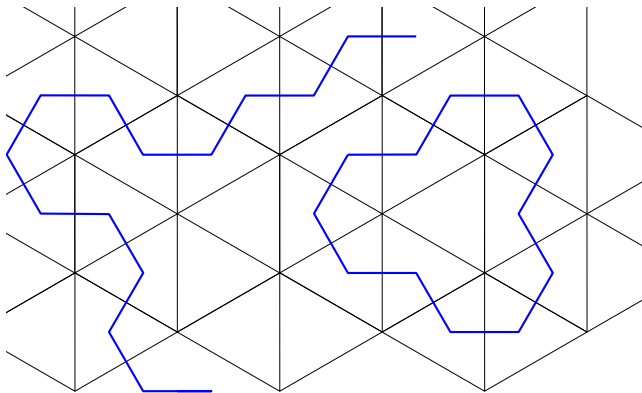


The weight is calculated in the same way:

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# $O(n)$ model on the dual lattice

Triangular lattice



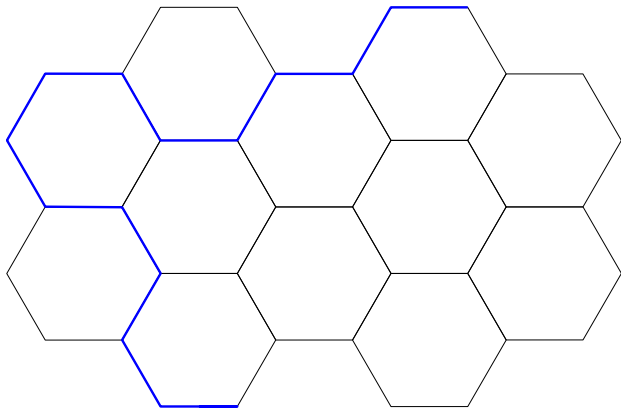
The weight of the configuration:

$$\omega(\text{conf}) = x^{\text{length}} \cdot n^{\text{loops}}$$



## $n = 0$ , self-avoiding walk

We pick boundary conditions with two vertices of degree 1 on the boundary and obtain the self-avoiding walk.

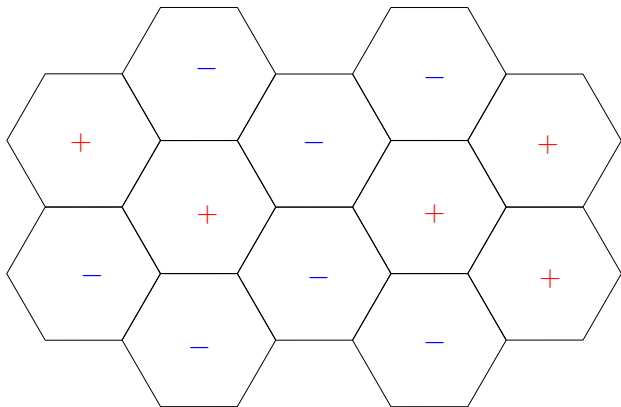


The weight of the configuration:

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## $n = 1$ , Ising model

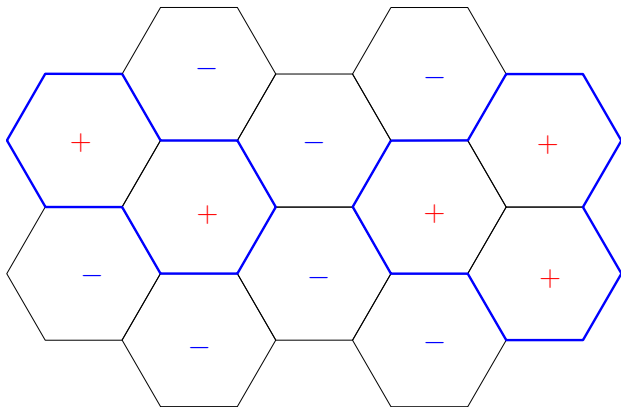
In this case, we do not count loops — Ising model. The configuration can be understood as walls between different adjacent spins.



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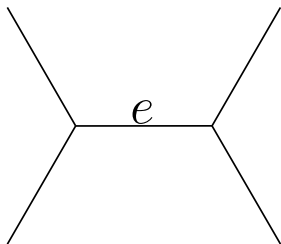
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## Observable on edges

From now on we will consider only the case of  $n \in [-2, 2]$ . We define the following real-valued function on the edges:

$$G(e) = \frac{1}{Z} \sum_{conf} \omega(conf) \cdot c_i,$$

where  $Z$  is the partition function and  $c_i$  depends on the configuration near the edge  $e$  and on the global loop structure.

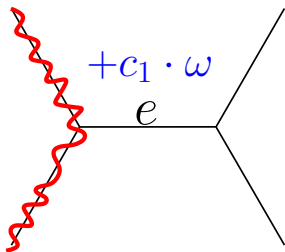


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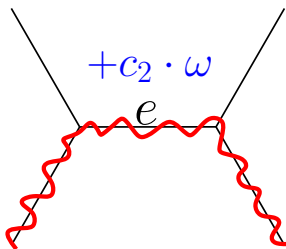


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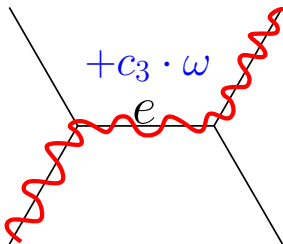


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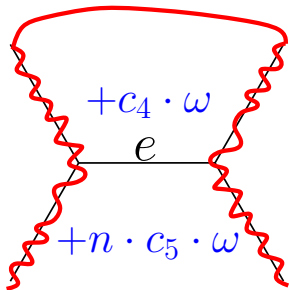


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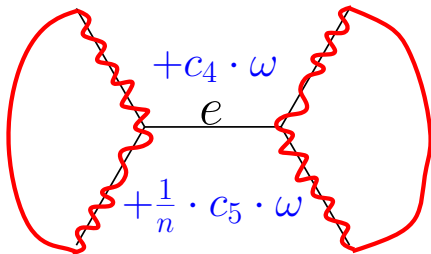


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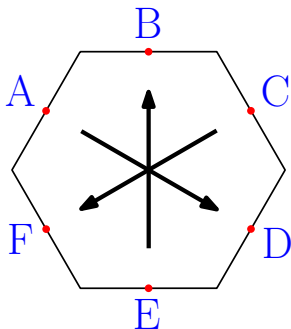
## Proposition

Q: Can we find such constants  $c_i$  that the observable  $G$  satisfies some local relations?

### Proposition

For each  $n$ , there are unique  $x, c_1, c_2, c_3, c_4, c_5$  such that for any hexagon

$$G(D) - G(A) = G(B) - G(E) = G(F) - G(C)$$



## formulae for $c_j$

### Remark

The relation is equivalent to the fact that  $G$  is real and the contour integral of  $G$  around each hexagon is 0:

$$G(D) + G(C)e^{\frac{\pi}{3}i} + G(B)e^{\frac{2\pi}{3}i} + G(A)e^{\frac{3\pi}{3}i} + G(F)e^{\frac{4\pi}{3}i} + G(E)e^{\frac{5\pi}{3}i} = 0$$

For each  $n$ , the special value of  $x$  is  $\frac{1}{\sqrt{2+\sqrt{2-n}}}$  — the critical value, unrigorously derived by Nienhuis. Coefficients are:

$$c_1 = n - 4 - 2\sqrt{2-n}$$

$$c_2 = 4 - n + 4\sqrt{2-n}$$

$$c_3 = 2 + 2\sqrt{2-n}$$

$$c_4 = 2n - 10 - 6\sqrt{2-n}$$

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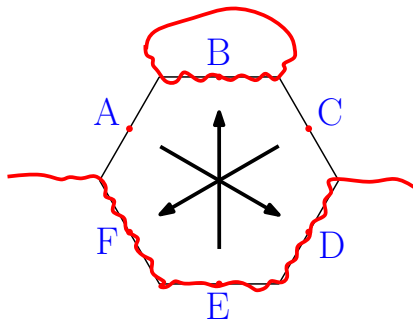
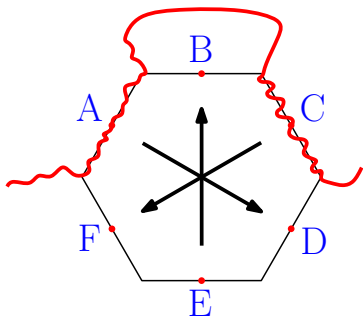
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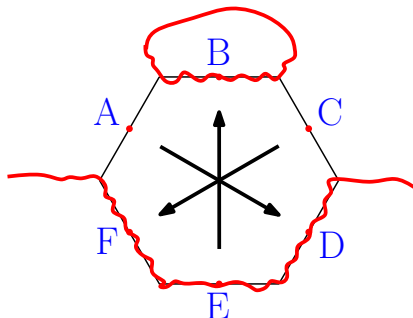
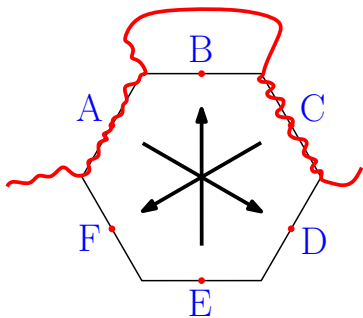
## Sketch of the proof, XOR



$$G(D) - G(A) : x^2(c_1 - c_2) + x^4 n(c_3 - c_4 - \frac{1}{n}c_5)$$

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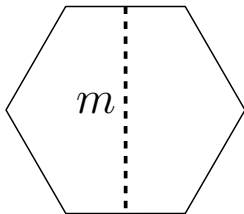
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## Observable on midlines of hexagons

We define the following real-valued observable:

$$H(m) = \frac{1}{Z} \sum_{conf} \omega(conf) \cdot d_i,$$

where  $Z$  is the partition function and  $d_i$  depends on the configuration near the midline  $m$  and on the global loop structure. This function  $H$  will be just a way to rewrite  $G$ .

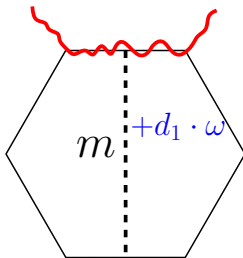


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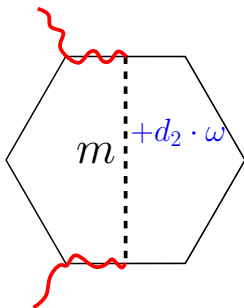


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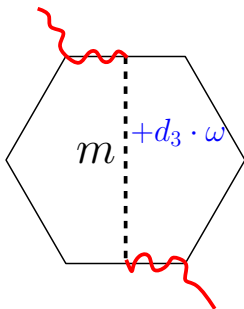


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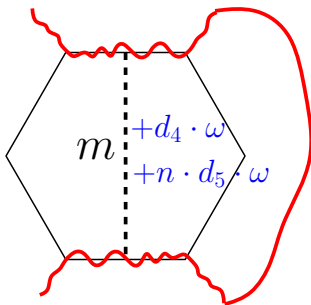


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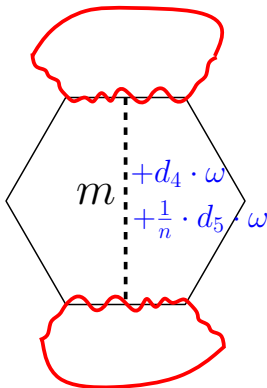


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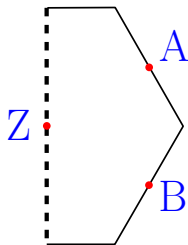
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### Proposition

For each  $n$ , there are unique  $d_1, d_2, d_3, d_4, d_5$  such that for any two adjacent edges

$$G(A) + G(B) + H(Z) = \text{constant}$$



## formulae for $d_i$

### Remark

For each  $n$ , the constant in the relation is equal to  $\frac{-2n}{1+\sqrt{2-n}}$ .

Coefficients are:

$$d_1 = \frac{10 - 5n - (n - 8)\sqrt{2 - n}}{1 + \sqrt{2 - n}}$$

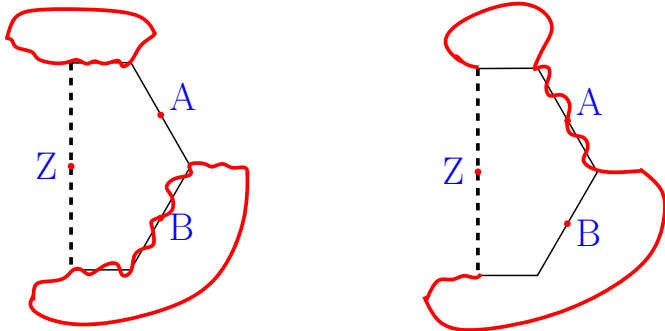
$$d_2 = \sqrt{2 + \sqrt{2 - n}} \cdot \frac{-4 - 2\sqrt{2 - n}}{1 + \sqrt{2 - n}}$$

$$d_3 = \frac{2n - 8 - 6\sqrt{2 - n}}{1 + \sqrt{2 - n}}$$

$$d_4 = \frac{-8n + 16 + (-2n + 12)\sqrt{2 - n}}{1 + \sqrt{2 - n}}$$

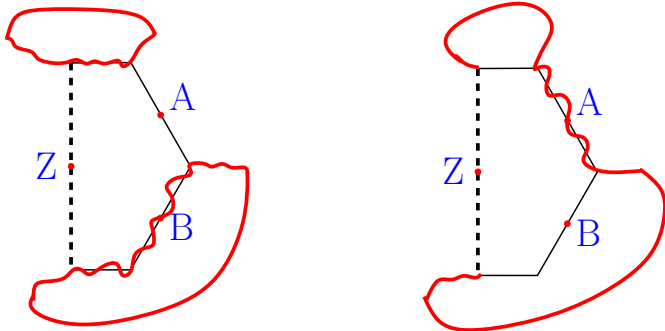
$$d_5 = \frac{4 + 2\sqrt{2 - n}}{1 + \sqrt{2 - n}}$$

## Sketch of the proof, semi-XOR



$$G(A) + G(B) + H(Z) : (c_4 + \frac{1}{n}c_5 + c_3 + d_4 + \frac{1}{n}d_5) + \frac{1}{xn}(c_2 + c_1 + d_2)$$
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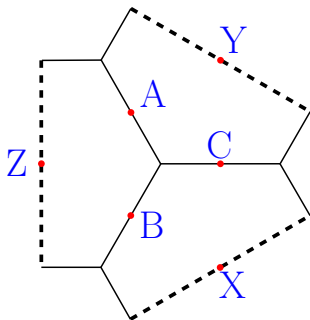
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## Connection between observables $G$ and $H$

Using the relation between  $G$  and  $H$ , one can rewrite  $G$  as a linear combination of  $H$ :

$$2G(C) = H(Z) - H(X) - H(Y) - \frac{2n}{1 + \sqrt{2-n}}$$



# $G$ and $H$ as deformations of the lattice

Local relations for observables  $G$  and  $H$  can be proved just combinatorially — by considering pairs of configurations. It looks like some miracle. A priori the number of equations on the coefficients is quite big, but finally we get just 6 different equations.

To explain this miracle and to get a conjecture for the limiting object, we will consider another approach to these observables. They can be obtained as the effect of infinitesimal non-flat deformations of the lattice.

This resembles the stress-energy tensor, which describes the effect of local transformations of the metric — inserting conformal anomalies.

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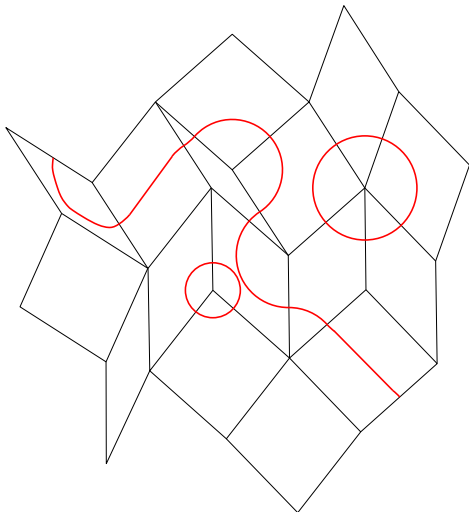
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## $O(n)$ model on a rhombic tiling

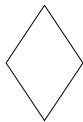
We consider the  $O(n)$  model on any rhombic tiling of some domain.

The weight of the configuration:

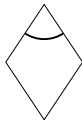
$$\omega(\text{conf}) = \prod_{r - \text{rhombus}} \omega(r) \cdot n^{\text{loops}}$$



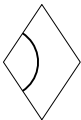
# Weight of a rhombus



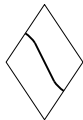
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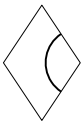
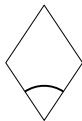
$u_1$



$u_2$



$v$



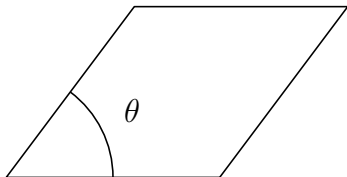
$w_1$



$w_2$

## Critical weights, historical remark

We will consider a specific family of weights parametrized by angle  $\theta$  of the rhombus. They were discovered as the weights, for which parafermionic observable satisfies half of Cauchy-Riemann equations. These weights satisfy also the Yang-Baxter equation.

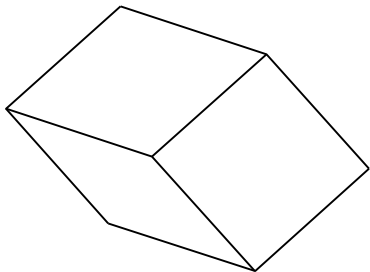
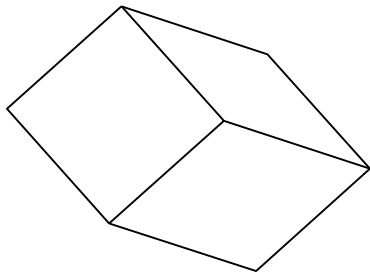


First integrable weights were discovered by Nienhuis in 1990 as the solutions to Yang-Baxter equation.

In 2009, Cardy and Ikhlef discovered the same weights as the weights for which the parafermionic observable satisfies some particular equation.

# Yang-Baxter equation

We fix the configuration outside the hexagon. Then the sum over all possible connections inside it is preserved under the following transformation:





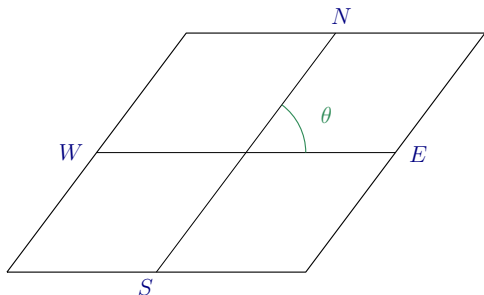
## Parafermionic observable

This observable was introduced by Smirnov. It is defined on edges and for an edge with the middle  $z$  it can be computed as:

$$F_a(z) = \sum_{conf} \omega(conf) e^{-i\sigma W(\gamma)},$$

where the sum is taken over the configurations with several loops and one walk  $\gamma$  from  $a$  to  $z$ ,  $a$  is the middle of some fixed edge on the boundary (the origin),  $\sigma$  — some parameter,  $W(\gamma)$  is the winding of  $\gamma$ .

# Cauchy-Riemann equation



**Lemma** For  $n = -2 \cos(-\frac{2\pi}{3}(2\sigma + 1))$  and the critical weights corresponding to the angle  $\theta$  we have, on any rhombus,

$$F_a(S) + e^{i\theta} F_a(E) - F_a(N) - e^{i\theta} F_a(W) = 0.$$

## Special values of $\sigma$ — Ising and SAW

If  $\sigma = \frac{1}{2}$ , then  $n = 1$  — the case of the Ising model. This (fermionic) observable and its modifications were used to establish conformal invariance of the Ising model (Smirnov et al).

If  $\sigma = \frac{5}{8}$ , then  $n = 0$  — the case of the self-avoiding walk. This observable was the main tool in the proof of Duminil-Copin and Smirnov for connectivity constant of hexagonal lattice.

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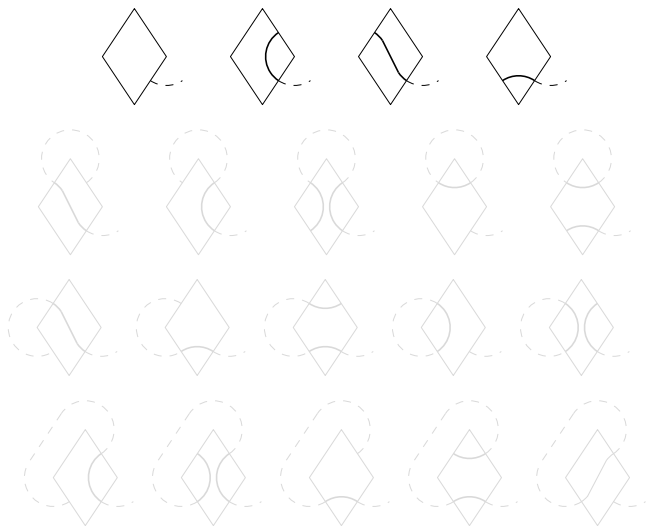
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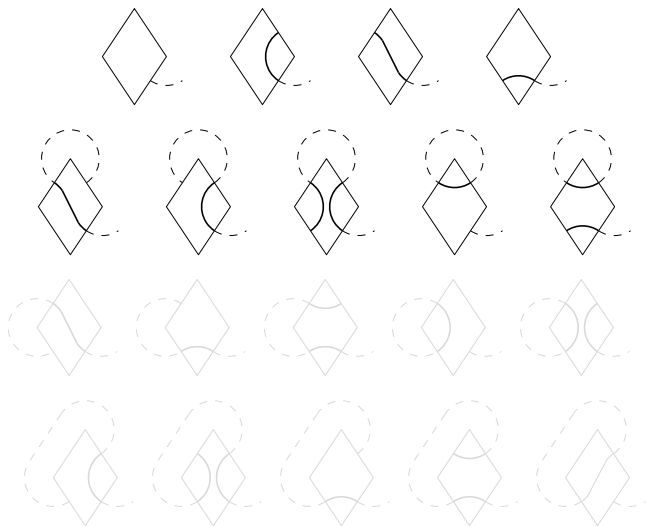
## Sketch of the proof of Lemma

Lemma can be proved by local transformations of the configuration inside the rhombus. There are four cases depending on the configuration outside the rhombus:



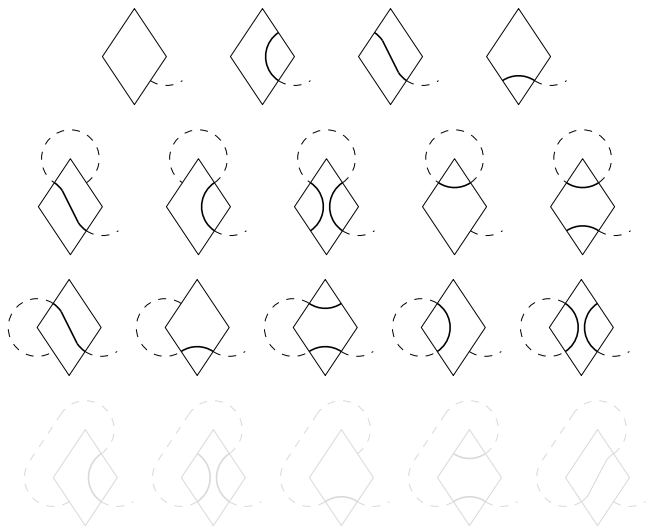
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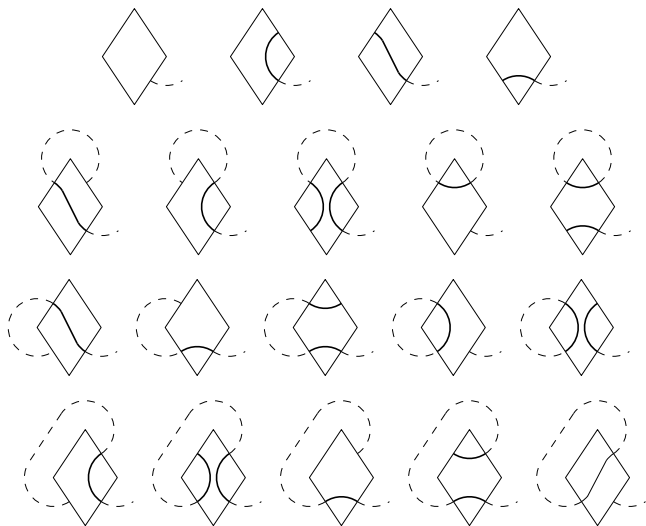
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## Critical weights, formulae

We take  $s = \sigma - 1$ ,  $n = -2 \cos \frac{4\pi}{3} s$ .

$$t = \frac{\sin^3 \frac{2\pi}{3} s}{\sin \frac{\pi}{3} s} + \sin \left( \theta - \frac{\pi}{3} \right) s \cdot \sin \left( \frac{2\pi}{3} - \theta \right) s$$

$$u_1 = \frac{1}{t} \cdot \sin \left( \pi - \theta \right) s \cdot \sin \frac{2\pi}{3} s$$

$$u_2 = \frac{1}{t} \cdot \sin \theta s \cdot \sin \frac{2\pi}{3} s$$

$$v = \frac{1}{t} \cdot \sin \theta s \cdot \sin \left( \pi - \theta \right) s$$

$$w_1 = \frac{1}{t} \cdot \sin \left( \frac{2\pi}{3} - \theta \right) s \cdot \sin \left( \pi - \theta \right) s$$

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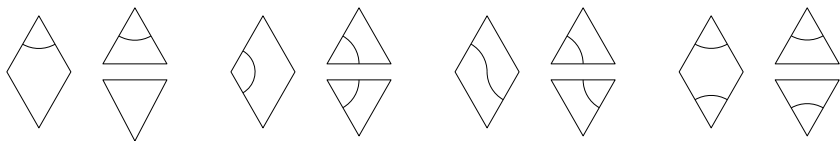
## The case $\theta = \frac{\pi}{3}$

For  $\theta \in (\frac{\pi}{3}, \frac{2\pi}{3})$ , the weights are positive.

For  $\theta = \frac{\pi}{3}$  the weights can be factorized, i.e.

$$u_2 = v = w_1 = (x_c)^2, \quad u_1 = x_c = \frac{1}{\sqrt{2 + \sqrt{2 - n}}} \quad \text{and} \quad w_2 = 0.$$

This is the critical  $O(n)$  model on the honeycomb lattice.



$$u_1 = x_c = \frac{1}{\sqrt{2 + \sqrt{2 - n}}} \quad u_2 = x_c^2$$

$$v = x_c^2$$

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## Remark about non-flat case

The lemma is proved only by the transformations inside the rhombus, we never use that our lattice is planar. Thus, we can consider the  $O(n)$  model on any set of rhombi with different angles and integrable weights on each of them.

We can add equilateral triangles. The weight of a triangle is either 0 if it is empty or  $x_c$  if it contains an arc of the configuration.

The important case for us — inserting conical singularities. This means that we allow the sum of the angles around some vertices of rhombi to be not equal to  $2\pi$ .

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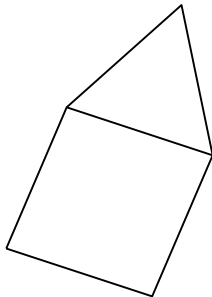
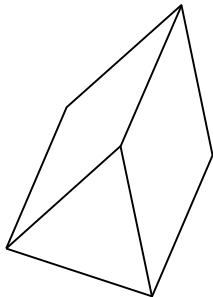
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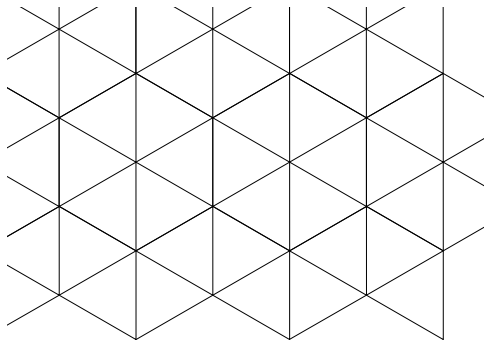
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## Pentagonal equation

We fix the configuration outside the pentagon. Then the sum over all possible connections inside it gets multiplied by some constant (depending only on  $n$ ) after the following transformation:



## Deformation of the lattice



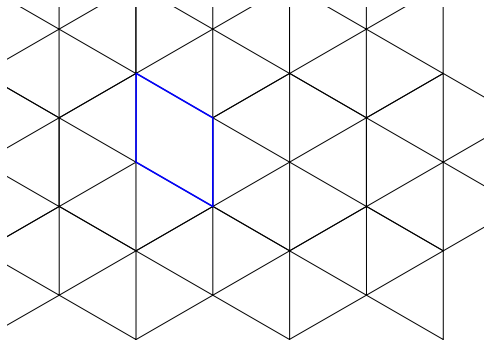
We consider the triangular lattice.

Then we pick any two adjacent triangles and join them into a rhombus.

And then we deform this rhombus.



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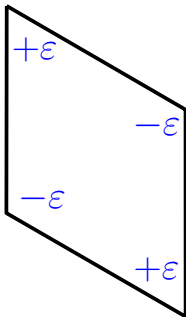


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## 1st game — constants for $G$



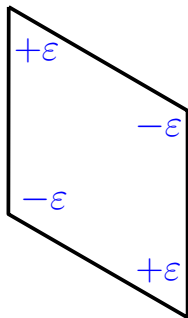
The XOR-observable is the derivative of weights after inserting these four singularities.

After such a deformation, we obtain another rhombus — with another integrable weights.

Hence, each configuration contributes:

$\omega(\text{conf}) \cdot \left\{ \text{the logarithmic derivative of its weight in this rhombus at } \frac{\pi}{33/44} \right\}$

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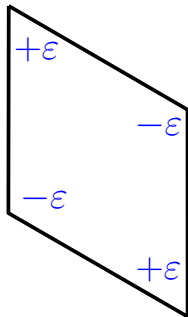
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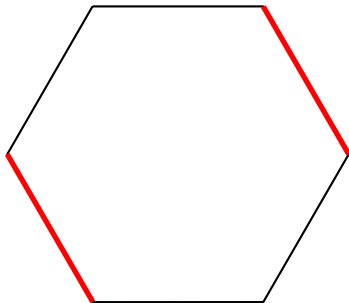
## Relation for the XOR-observable

These will be exactly our coefficients  $c_1, \dots, c_5$ .

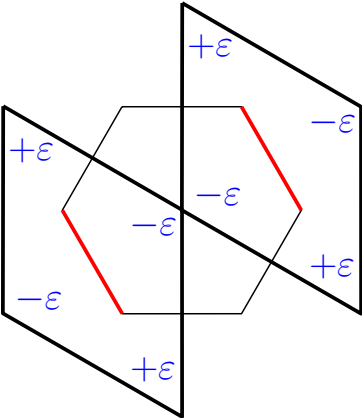
Q: How do we get the relation?

We should just rewrite everything in terms of singularities.

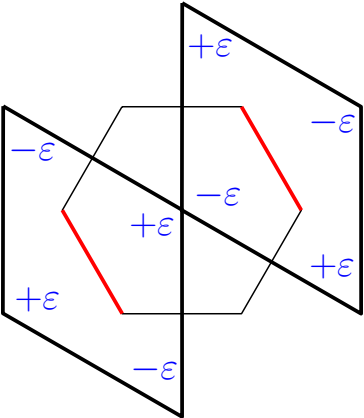
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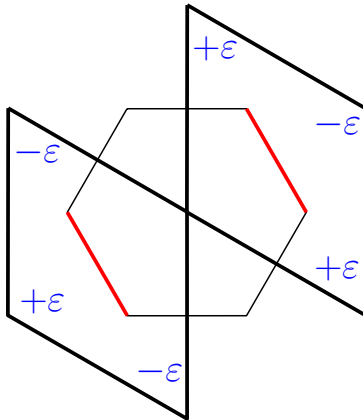


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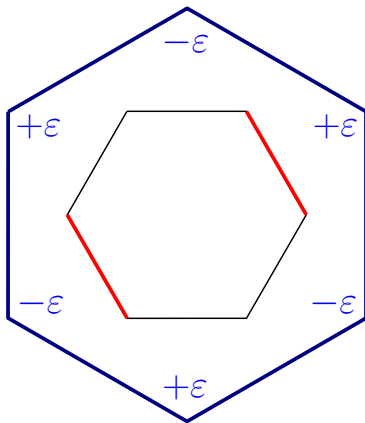




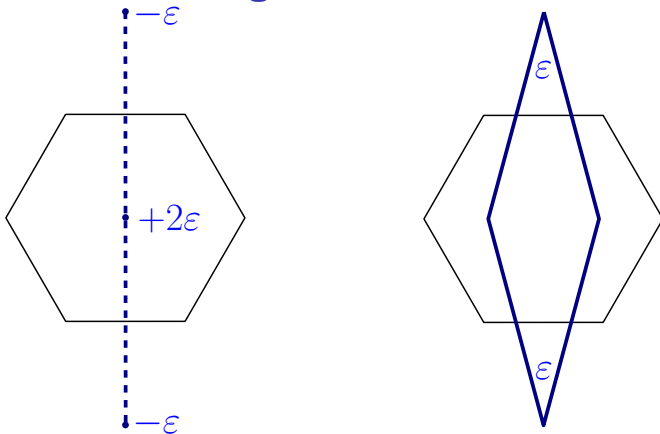
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## 2nd game — constants for $H$

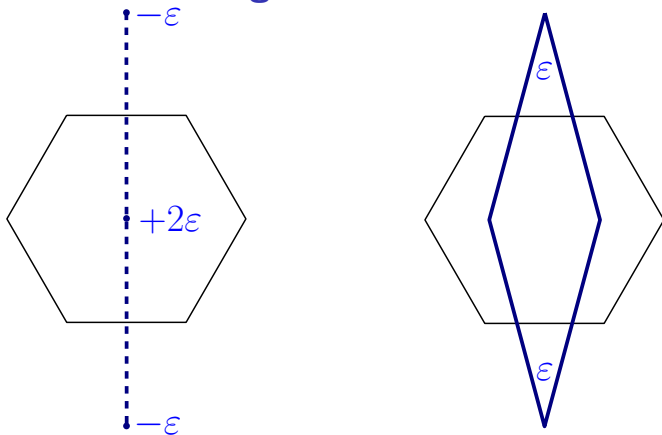


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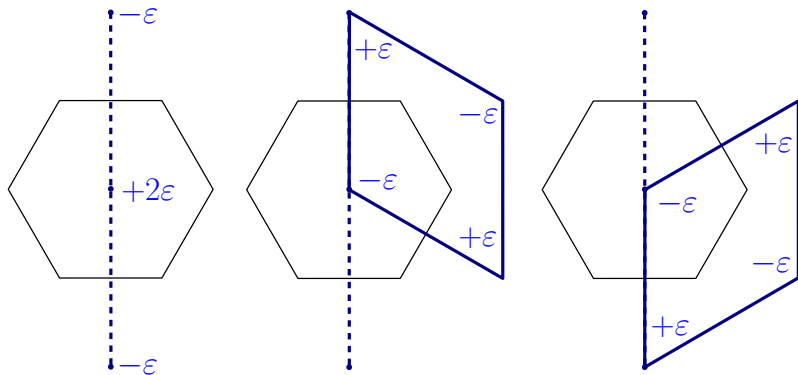


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## Relation for the semi-XOR-observable



$$H(X) + G(A) + G(B) = -\frac{2n}{1+\sqrt{2-n}}$$

## Ising case

$$d_1 = 6$$

$$d_2 = -6 \cdot \frac{1}{\sqrt{3}}$$

$$d_3 = -6$$

$$d_4 = 9$$

$$d_5 = 3$$

The coefficient for visiting 2 edges is  $9 + 3 = 12$ , and the coefficient for visiting one edge is 6, so two times smaller.

Because of this one can represent  $H$  (and hence  $G$ ) as a linear combination of the values of the fermionic observable.

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## Theorem

Let  $\Omega$  be a bounded planar simply connected domain and  $\varphi : \Omega \rightarrow \mathbb{D}$  be a conformal map. Then for the plus boundary conditions

$$\lim_{\delta \rightarrow 0} G(z)\delta^{-2} = \text{const} \cdot \text{Re}(S\varphi(z)),$$

where  $S\varphi(z) = \frac{\varphi'''(z)}{\varphi'(z)} - \frac{3}{2} \left( \frac{\varphi''(z)}{\varphi'(z)} \right)^2$  — the Schwarzian derivative of  $\varphi$ .

**first level completed**  
**level up**  
**next level — Big Boss**