Absence of phase transitions in 2D O(N) Heisenberg models (Analysis of the RG Flow)

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Millenium Problem

(1) Construct 4D Non-Abelian Gauge Theory (via LGT)
(2) Prove or disprove Quark Confinement and Asymptotic Freedom
(3) Show Mass generations in 2D σ models.

Difficulties in 4D LGT/ 2D σ Models

1. we need non-perturbative calculations since we have to show these claims for all coupling constants
2. we need good definitions of block spins and domain walls for non abelian spin systems and lattice gauge systems
3. strong non-linearlity prohibits block spin transformation
4. perhaps, very complicated steps of block spin transformations (assuming the existence)

History of 2D Spin Systems

1. 2D Ising spin, existence of spontaneous magnetization, R.Peierls (1936), L.Onsager (1944)
2. Kosterlitz-Thouless Transition in 2D XY model, J.Fröhlich and T.Spencer (1982)
3. non-existence of phase transition in the Heisenberg model with large N (this talk)

Use RG method of WD-GK type

The 2D $O(N)$ Heisenberg model. Strong non-linerity:

$$< \cdots >= \int (\cdots) \exp[\sum_{n.n.} \phi_x \phi_y] \prod_x \delta(\phi^2(x) - N\beta) d^N \phi_x,$$

Introduce auxiliary fields $\{\psi(x); x \in Z^2\}$ to keep $\phi(x) \in R^N$ on S^{N-1} with radius $\sqrt{N\beta}$:

$$\begin{aligned}
&\prod_x \delta(\phi^2(x) - N\beta) \\
&\quad = \int \exp[iN^{-1/2} \sum_x (\phi^2(x) - N\beta)\psi(x)] \prod \frac{d\psi(x)}{2\pi} \\
&\quad = \int \exp[iN^{-1/2} <: \phi^2 :, \psi >] \prod d\psi(x)
\end{aligned}$$

The Yukawa model with imaginary coupling constant W_0.

$$\int (\cdots) \exp[-W_0(\phi,\psi)] \prod_x d^N\phi(x) d\psi(x)$$

$$W_0 = \frac{1}{2} < \phi, (-\Delta + m_0^2)\phi > - \frac{i}{\sqrt{N}} <: \phi^2 :, \psi >$$

$$: \phi^2 : (x) = \sum_{i=1}^{N} \phi_i^2(x) - NG(0), \quad \beta = G(0)$$

Here $G(0) = \beta$ means $m_0^2 \sim 32 e^{-4\pi\beta}$:

$$G(x) = \frac{1}{-\Delta + m_0^2}(x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{e^{ipx}}{m_0^2 + 2\sum(1-\cos p_i)} \prod \frac{dp_i}{2\pi}$$

Integrate $\exp[-W_0]$ over high-momentum parts ($|p_i| > L^{-1}$) $\phi_>$ and $\psi_>$ by putting

$$\psi(x) \to \psi_>(x) + \psi_<(x), \quad \phi(x) \to \phi_>(x) + \phi_<(x)$$

Replace Lx by x

$$\phi_<(Lx) = \phi(x), \quad \psi_<(Lx) = \psi(x)$$

Low momentum parts are again on Z^2 by scaling $Lx \to x$.

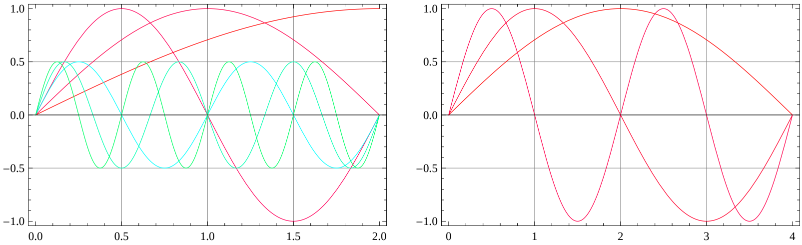

Figure: Integrate out High Mom. part ($\phi_> = \xi$). Then Scale down by 1/2: $\phi(x) = \phi_<(x) + \phi_> \rightarrow \phi_<(x) = \phi(x/L)$

Repeat n times and show new system is in high temp. region:

$$W_n \sim \frac{1}{2} < \phi, (-\Delta)\phi > - \frac{i}{\sqrt{N}} < (\phi^2 - N\beta_n), \psi > + < \psi, \psi >$$

and $\beta_n = O(1)$. Integrate by ψ. Disappearance of double well potential. Then we have reached the massive system:

$$V_n \sim \frac{1}{2} < \phi, (-\Delta)\phi > + \frac{1}{N} \sum_x (\phi^2(x) - N\beta_n)^2$$

$$\beta_n = 1$$

Use BST to get W_{n+1} from W_n:

Mathematically Controlable BST:

$$\exp[-W_{n+1}(\{\phi_{n+1},\psi_{n+1}\})]$$
$$\equiv \int \exp[-W_n(\{\phi_n,\psi_n\})] \prod_{x\in\Lambda_{n+1}} \delta(\phi_{n+1}(x)-(C\phi_n)(x))$$
$$\times\delta(\psi_{n+1}(x)-(C'\psi_n)(x)) \prod_{\zeta\in\Lambda_n} d\phi_n(\zeta)d\psi_n(\zeta)$$

where $\Lambda_n = \{L^{-n}\Lambda\cap Z^2\}$ and

$$\phi_1(x) = (C\phi)(x) = L^{-2} \sum_{-L/2<\zeta_i\leq L/2} \phi(Lx+\zeta)$$
$$\psi_1(x) = (C'\psi)(x) = \sum_{-L/2<\zeta_i\leq L/2} \psi(Lx+\zeta)$$

Why do we choose these C and C' ? See the dimensions.

$$[\phi] = (\text{length})^0, \quad [\psi] = (\text{length})^{-2}$$

and C and C' leave the fundamental Gaussian measures invariant

$$\exp[- < \phi, G^{-1}\phi >] \prod d\phi(x)$$
$$\exp[- < \psi, G^{\circ 2}\psi >] \prod d\psi(x)$$

and

$$G^{-1} = -\Delta + m_0^2, \quad G^{\circ 2}(x,y) = G(x,y)^2$$

Why so for ϕ ?

Expect self-similarity

$$< \phi_{n+1}(x)\phi_{n+1}(y) >_0 \sim < \phi(x)\phi(y) >_0 = \frac{1}{|x-y|^{D-2}} \qquad (1)$$

where $< f(\phi) >_0 = \int f(\phi) e^{-(1/2)<\phi,(-\Delta)\phi>} \prod_x d\phi(x)$ We take average for $D = 2$

$$L^{-4} \sum_{\zeta,\xi\in\Delta} \log|Lx+\zeta-Ly+\xi| = \log|x-y|$$

Why so for ψ ?

First $[\psi] = (\text{length})^{-2}$. Then $C' = L^2 C$ for BST. Cruedly

$$\langle \psi(x), \psi(y) \rangle = [G^{\circ 2}]^{-1}(x, y) \sim \frac{1}{|x-y|^4}$$

Then this measure is left invariant by $C' = L^2 C$.

$$\sum_{\zeta, \xi \in \Delta} \frac{1}{|Lx + \zeta - Ly - \xi|^4} = \frac{1}{|x-y|^4}$$

Cooresponding to the decomposiiton

$$\langle \phi_n, G_n^{-1} \phi_n \rangle = \langle \phi_{n+1}, G_{n+1}^{-1} \phi_{n+1} \rangle + \langle \xi_n, Q^{+} G_n^{-1} Q \xi_n \rangle$$

$$\begin{aligned} \phi_n(x) &= \text{Next order BS} + \text{Zero Ave. Fluct.} \\ &= \sum_y A_{n+1}(x,y) \phi_{n+1}(y) + \sum_y (Q)_{xy} \zeta_y \end{aligned}$$

Approximately,

$A \in \mathrm{Mat}(L^2, 1)$, $Q \in \mathrm{Mat}(L^2, L^2 - 1)$:

$$A(x,y) \sim \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, Q(x,y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Q^{+} acts as a differential operator.

Explicitly

$$\begin{aligned}
\phi_n(x) &= A_n\phi_{n+1} + Q\xi(x) \\
&\sim \phi_{n+1}([x/L]) + Q\xi(x) \\
\psi_n(x) &= \tilde{A}_n\psi_{n+1} + Q\tilde{\psi}(x) \\
&\sim \frac{1}{L^2}\psi_{n+1}([x/L]) + Q\tilde{\psi}(x)
\end{aligned}$$

are the block spins at the distance scale L^{n+1}.

$$\sum_x \phi_n^2(x) \sim L^2 \sum_x \phi_{n+1}^2(x), \quad \sum_x \psi_n^2(x) \sim L^{-2} \sum_x \psi_{n+1}^2(x)$$

All terms in this model are marginal or irrelevant!!

Theorem on the RG flow

The main part of W_n is represented by 4 terms and three parameters m_n^2, γ_n and β_n:

$$W_n(\phi_n, \psi_n) = \frac{1}{2} < \phi_n, G_n^{-1} \phi_n > + \frac{1}{2} \gamma_n < \partial_\mu \phi_n, (\phi_n \otimes \phi_n) \partial_\mu \phi_n > + < \psi_n, H_n^{-1} \psi_n > + \frac{i}{\sqrt{N}} <: \phi_n^2 :_{G_n}, \psi_n >$$

where

1. $G_n^{-1} = -\Delta + m_n^2$, $m_n^2 = L^{2n} m_0^2$
2. $: \phi_n^2(x) := \phi_n^2(x) - N G_n(0)$, $G_n(0) = \beta_n = \beta_{n-1} - O(1)$.
3. $\gamma_n = \text{const} \frac{n}{N} > 0$.
4. $H_n = O(1) > 0$

and

$$\frac{1}{4}\sum_{x}\sum_{\mu}(\partial_\mu \phi_n^2(x))^2 = \langle \partial_\mu \phi_n, (\phi_n \otimes \phi_n)\partial_\mu \phi_n \rangle + \text{(irrel.)}$$

The flow is described by three parameters

$$\begin{aligned} m_n^2 &= L^{2n} m_0^2 \sim \exp[-4\pi\beta + 2n \log L] \to O(1), \\ \beta_n &= \beta - \mathrm{const.} n \to O(1) \\ \gamma_n &= n/N \quad (\log \beta < n < O(\beta)) \end{aligned}$$

except for large and/or domain wall field region.
So the absence of phase transition follows from this theorem (large and/or domain wall field regions cost energy.)

This means absence of phase transitions:

Take n large so that $\beta_n = O(1)$. Integrate $\exp[-W_n]$ by $d\psi_n$

$$\exp[-V(\phi_n)] = \int \exp[-W_n(\phi_n, \psi_n)] \prod d\psi_n$$

Then

$$V_n(\phi_n) = \frac{1}{2} < \phi_n, (-\Delta + m_n^2)\phi_n >$$
$$+ \frac{\beta}{N} \sum (\phi_n^2(x + e_\mu) - \phi_n^2(x))^2 + \frac{1}{N} \sum (\phi_n^2(x) - N\beta_n)^2$$

which is massive no matter how large γ_n is, since $\beta_n = O(1)$ for large n .

Sketch of the Proof

1st step: We set $\phi_n = A_{n+1}\phi_{n+1} + Q\xi_n$ so that

$$
\begin{aligned}
< \phi_n, G_n^{-1}\phi_n > &= < \phi_{n+1}, G_{n+1}^{-1}\phi_{n+1} > + < \xi_n, \Gamma_n^{-1}\xi_n >, \\
\Gamma_n^{-1} &= Q^{+}(-\Delta)Q > O(1)
\end{aligned}
$$

If $\phi_n(x)$ changes slowly in x

$$
\begin{aligned}
&< \partial_\mu\phi_n, (\phi_n \otimes \phi_n)\partial_\mu\phi_n > \\
&\quad \to < \partial_\mu\phi_{n+1}, (\phi_{n+1} \otimes \phi_{n+1})\partial_\mu\phi_{n+1} > \\
&\qquad + < \partial_\mu Q\xi_n, (\phi_{n+1} \otimes \phi_{n+1})\partial_\mu Q\xi_n > \\
&\quad \to < \partial_\mu\phi_{n+1}, (\phi_{n+1} \otimes \phi_{n+1})\partial_\mu\phi_{n+1} > \\
&\qquad + < \xi_n, \Gamma_n^{-1}(\phi_{n+1} \otimes \phi_{n+1})\xi_n >
\end{aligned}
$$

The ξ_n term (the fluctuation) is a
perturbed Gaussian of Matrix form:

$$\int \exp\left[-\frac{1}{2} < \xi_n, P\xi_n > -\frac{2i}{\sqrt{N}} < Q^+(\phi_{n+1}\psi), \xi_n >\right] d\xi_n$$

$$P = (\Gamma_n^{-1} + i\alpha Q^+\psi_n Q) \otimes 1_N + \gamma_n \Gamma_n^{-1} \circ (\phi_n \otimes \phi_n)$$

$$\Gamma_n = [Q^+(-\Delta + m_n^2)Q]^{-1} \quad (\geq O(1) > 0, \text{ short range})$$

where $[\Gamma_n^{-1} \circ (\phi_n \otimes \phi_n)](x, y)$ is an $N \times N$ matrix:

$$(\Gamma^{-1} \circ (\phi \otimes \phi))_{i,x,j,y} = (\Gamma)^{-1}(x, y)\phi_i(x)\phi_j(y)$$

The lattice Laplacian P

$$P = (\Gamma_n^{-1} + i\alpha Q^{+}\psi_n Q) \otimes 1_N + \gamma_n \Gamma_n^{-1} \circ (\phi_n \otimes \phi_n)$$

depends on spins ϕ_n through the projection $\phi_n \otimes \phi_n$. We can define

$$P^{-1} = \left[(\Gamma_n^{-1} + i\alpha Q^{+}\psi_n Q) \otimes 1_N + \gamma_n \Gamma_n^{-1} \circ (\phi_n \otimes \phi_n) \right]^{-1}$$

and integrate over ξ_n only for outside of the domain walls.

What is the most proper definition of the domain walls ?

Domain walls are paved set such that

$$|\phi_n(x)\phi_n(y) - N\beta_n| > N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w, \exists y \in D_w$$

$1/2$ is the central limit theorem for $\sum : \xi_i^2 :$. Outside of D_w,

$$|\phi_n(x)\phi_n(y) - N\beta_n| < N^{1/2+\varepsilon}\exp[(c/10)|x-y|]$$

$$\forall x \in D_w^c, \forall y \in D_w^c$$

Outside of the domain walls

$$\begin{aligned} (\Gamma_n^{-1} \circ (\phi_n \otimes \phi_n))_{x,i,y,j} &\equiv \Gamma_n^{-1}(x,y)\phi_{n,i}(x)\phi_{n,j}(y) \\ &\sim \Gamma_n^{-1}(x,y)\phi_{n,i}(x)\phi_{n,j}(x) \\ \phi_n(x)\phi_n(y) &= NG_n(x,y) + :\phi_n(x)\phi_n(y): \end{aligned}$$

and Wick ordered terms are smaller than the Wick Const.!.
The integration yields

$$\det{}^{-1/2}(P) \exp\left[-\frac{1}{N} < \psi_n, [\phi_{n+1} Q^{+} \frac{1}{P} Q \phi_{n+1}] \psi_n > \right]$$

$$\begin{aligned} \det{}^{-1/2}(P) &= \det{}^{-(N-1)/2}(\Gamma_n^{-1} + i\alpha Q^{+}\psi Q) \\ &\quad \times \det{}^{-1/2}(\Gamma_n^{-1} + i\alpha Q^{+}\psi Q + \gamma_n ||\phi_n||^2) \end{aligned}$$

2nd step: Expand the determinant up to the second order:

1. Thus expand the determinant up ψ^2:

$$\det{}^{-N/2}\left(1+\frac{2i}{\sqrt{N}}\Gamma_n\psi_n\right)=\det{}_3^{-N/2}\left(1+\frac{2i}{\sqrt{N}}\Gamma_n\psi_n\right)$$

$$\times\exp[-i\sqrt{N}\mathrm{Tr}\Gamma_n\psi_n-\mathrm{Tr}(\Gamma_n\psi_n)^2]$$

2. The yukawa term $\exp[-i<\phi_n^2,\psi_n>/\sqrt{N}]$ does not yield $\exp[-O(\psi_n^2)]$ if γ_n is large
3. (1)+(2) yield the Gaussian integral:

$$\exp[-<\psi_n,[H_n+\Gamma_n^{\circ 2}]\psi_n>]$$

1. the 1st term

$$i\sqrt{N}\mathrm{Tr}(\Gamma_n\psi_n)$$

shifts β_n by $O(1)$ since

$$(i/\sqrt{N})\langle\phi_n^2 - N(\beta_n - \Gamma_n), \psi_n\rangle, \quad \beta_n - \Gamma_n(0) = \beta_{n+1}$$

2. the 2nd term $\sim \mathrm{Tr}(\Gamma_n\psi_n)^2$ forms a strictly posotive part of the Hamiltonian

$$< \psi_n, H_n^{-1}\psi_n > \text{ of } \psi_n$$

For$(x, y) \in \Lambda^{\otimes 2}$, $\dfrac{1}{P}(x, y) \in \mathrm{Mat}(N, N)$ and

$$(\phi_{n+1} \frac{1}{P} \phi_{n+1})(x, y) = \sum_{i,j} \phi_{n+1,i}(x) \left(\frac{1}{P}\right)_{ix;jy} \phi_{n+1,j}(y)$$

$$= ||\phi_{n+1}||^2 \frac{1}{\Gamma_n^{-1} + i\alpha Q^+ \psi_{n+1} Q + \gamma_n ||\phi_{n+1}||^2}(x, y)$$

extracts the orientation of ϕ_{n+1} from $\dfrac{1}{P}$.

If $\gamma_n > O(1)$, the γ_n terms has no effects on the flow.

$$\phi_{n+1} \frac{1}{P} \phi_{n+1} \sim \begin{cases} ||\phi_{n+1}||^2 \Gamma_n & \text{if } \gamma_n << 1 \\ \gamma_n^{-1} & \text{if } \gamma_n >> 1 \end{cases}$$

3rd step:

Decompose $< \psi_n, H_n^{-1} \psi_n >$ into block spins ψ_{n+1} and fluctuations $\tilde{\psi}_n$. Integrate by $\tilde{\psi}_n$.

$$< \psi_n, H_n^{-1} \psi_n > = < \psi_{n+1}, H_n^{-1} \psi_{n+1} > + < \tilde{\psi}_n, Q^+ H_n^{-1} Q \tilde{\psi}_n >$$

We obtain the additional contribution to γ_n from through

$$\frac{i}{\sqrt{N}} \langle : \phi_{n+1}^2 :_{G_n}, Q\tilde{\psi}_n \rangle$$

1. Again set $\psi_n = \tilde{A}_{n+1}\psi_{n+1} + Q\tilde{\psi}_n$ so that

$$\begin{aligned} &\exp[- < \psi_n, [H_n^{-1} + \Gamma_n^{\circ 2}]\psi_n >] \\ &= \exp[- < \psi_{n+1}, H_{n+1}^{-1}\psi_{n+1} > \\ &\quad - < \tilde{\psi}_n, Q^{+}[H_n^{-1} + \Gamma_n^{\circ 2}]Q\tilde{\psi}_n >] \end{aligned}$$

2. In the above, $(\tilde{A}_{n+1}f)(x) = L^{-2}\sum_{\zeta} f(Lx+\zeta)$, and Γ_n are exponentially decresing. So H_n^{-1} is always $O(1)$.
3. The integ. over $\tilde{\psi}$ yields small correcions to
$< \phi_{n+1}, (-\Delta)\phi_{n+1} >$
$< \partial_\mu\phi_{n+1}, [1 + \gamma_n\varphi_{n+1} \otimes \varphi_{n+1}]\,\partial_\mu\varphi_{n+1} >$

4th step: Closed recursion relations.

$$\exp[-\frac{1}{2} < \phi_n, G_n^{-1}\phi_n > -\frac{1}{2}\gamma_n < \partial_\mu \phi_n, (\phi_n \otimes \phi_n)\partial_\mu \phi_n >$$
$$- < \psi_n, H_n^{-1}\psi_n > - \frac{i}{\sqrt{N}} <: \phi_n^2 :_{G_n}, \psi_n >]$$
$$\times \left(\text{irrelevant terms like } \prod \det{}_3^{-N/2} \left(1 + \frac{2i}{\sqrt{N}} Q \Gamma_k Q^+ A \psi_n \right) \right)$$

where

$$G_n^{-1} \sim -\Delta + L^{2n} m_0^2, \quad H_n = O(1) > 0$$

This establishes non-existence of phase transitions !!

Thank you very much for your attension and patience!