Quantum team logic and Bell's Inequalities

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Traditional semantics

- Propositional logic
- Atoms: "Gate is open", "Red light is on", "Spin at 60° is up"
- True/false
- Proposition symbols $p_0, \ldots, p_n$
- Connectives $\neg$, $\wedge$, $\vee$, $\to$, e.g. $(p_0 \wedge p_1) \vee (\neg p_0 \wedge \neg p_1)$
- Valuation $v : \{p_0, \ldots, p_n\} \to \{0, 1\}$
- Valuation v models (a version of) truth
- Truth-value $v(\phi) \in \{0, 1\}$

Team semantics

- Team = a set X of valuations $v : \{p_0, \ldots, p_n\} \to \{0, 1\}$
- X models (a version of) "uncertainty" about $p_0, ..., p_n$
- X models "observations" about $p_0, ..., p_n$
- Can give meaning to "dependence" of p_i on p_j.
- Can give meaning to "independence" of p_i of p_j.

p_0	p_1	p_2	p_3
1	1	0	1
1	1	1	1
1	1	1	0
0	0	1	1
0	0	0	0

Multi-teams

- Multi-team = an indexed set X of valuations

$$v : \{p_0, \dots, p_n\} \to \{0, 1\}$$

(repetitions allowed)

- Notation: $dom(X) = \{p_0, \dots, p_n\}$.
- X models "probability" of different versions of truth about $\langle p_0, ..., p_n \rangle$
- X models "observations" about $\langle p_0, ..., p_n \rangle$
- Can give meaning to "probability"

$$[\phi]_X$$

of ϕ in X: the number of valuations satisfying ϕ divided by the size of $|X|$ (in the finite case).

- Examples $[\phi \wedge \neg\phi]_X = 0$, $[\phi \vee \neg\phi]_X = 1$.

	p_0	p_1	p_2	p_3
0	1	1	0	1
1	1	1	1	1
2	1	1	1	1
3	1	1	1	0
4	0	0	1	1
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0

A logic for arguing about probabilities

- We build a formal language for reasoning about statements of the form "the probability of ϕ (in the multi-team) is at least 0.125".
- We shorten this to $\phi \geq 0.125$.
- We allow also more general statements $a_0\phi_0 + \ldots + a_k\phi_k \geq c$, where the a_i are rationals.

A logic PTL for reasoning about probabilities

Suppose $\phi_0, \ldots, \phi_k$ are propositional formulas, $(a_j)_{j \leq k} \in \mathbb{Z}^k$ and $c \in \mathbb{Z}$, then

$$a_0\phi_0 + \ldots + a_k\phi_k \geq c$$

is an atomic formula of PTL.

The set of formulas of PTL is defined as follows:

- Atomic formulas are formulas;
- If α is a formula, then $\neg\alpha$ is a formula;
- If α and β are formulas, then $\alpha \wedge \beta$ is a formula.

Semantics

Suppose X is a multi-team and α a formula of PTL with propositional symbols in $dom(X)$. We define by induction on α the relation

$$\text{"}X \text{ satisfies } \alpha\text{", in symbols } X \models \alpha,$$

in the following way:

- $X \models a_0\phi_0 + \ldots + a_{k-1}\phi_{k-1} \geq c$ iff $a_0[\phi_0]_X + \ldots + a_{k-1}[\phi_{k-1}]_X \geq c$;
- $X \models \neg\alpha$ iff $X \not\models \alpha$;
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.

Deductive system

Propositional reasoning

A) All instances of propositional tautologies.

B) If $\alpha \to \beta$ and α, then β (modus ponens).

Probabilistic reasoning

C) $\phi \geq 0$.

D) $\phi \vee \neg\phi = 1$.

E) $\phi \wedge \psi + \phi \wedge \neg\psi = \phi$ (additivity).

F) If $\phi \equiv \psi$ in propositional logic, then $\phi = \psi$.

Linear inequalities

G) $\phi \geq \phi$.

H) $\sum_{j<k} a_j\phi_j \geq c \Leftrightarrow \sum_{j<k} a_j\phi_j + 0\psi \geq c$.

I) $\sum_{j<k} a_j\phi_j \geq c \Leftrightarrow \sum_{j<k} a_{\sigma(j)}\phi_{\sigma(j)} \geq c$ (for σ permutation on k).

J) $\sum_{j<k} a_j\phi_j \geq c \wedge \sum_{j<k} b_j\phi_j \geq d \Leftrightarrow \sum_{j<k}(a_j + b_j)\phi_j \geq c + d$.

K) $\sum_{j<k} a_j\phi_j \geq c \Leftrightarrow \sum_{j<k} da_j\phi_j \geq c$ (for $d > 0$).

L) $\sum_{j<k} a_j\phi_j \geq c \vee \sum_{j<k} a_j\phi_j \leq c$.

M) $\sum_{j<k} a_j\phi_j \geq c \Rightarrow \sum_{j<k} a_j\phi_j > d$ (for $c > d$).

Fagin-Halpern-Megiddo 1990

Theorem (Completeness)

Let α *be a formula of* PTL*. Then* ϕ *is provable in* PTL *if and only if* ϕ *is true in every multi-team.*

Logical Bell Inequalities

$$
\begin{aligned}
1 - [\textstyle\bigwedge_{j<k} \phi_j]_X &= [\textstyle\bigvee_{j<k} \neg\phi_j]_X \\
&= P(\left\{ i \in X \mid \tau(i)(\textstyle\bigvee_{j<k} \neg\phi_j) = 1 \right\}) \\
&= P(\textstyle\bigcup_{j<k} \{i \in X \mid \tau(i)(\neg\phi_j) = 1\}) \\
&\leq \textstyle\sum_{j<k} P(\{i \in X \mid \tau(i)(\neg\phi_j) = 1\}) \\
&= \textstyle\sum_{j<k} [\neg\phi_j]_X \\
&= \textstyle\sum_{j<k} (1 - [\phi_j]_X) \\
&= k - \textstyle\sum_{j<k} [\phi_j]_X.
\end{aligned}
$$

Hence

$$
\sum_{j<k} [\phi_j]_X \leq k - 1 + [\bigwedge_{j<k} \phi_j]_X. \tag{1}
$$

If the formula $\bigwedge_{j<k} \phi_j$ is contradictory (in the sense of propositional logic), then $[\bigwedge_{j<k} \phi_j]_X = 0$. Thus, the inequality (1) becomes

$$\sum_{j<k} [\phi_j]_X \leq k - 1. \tag{2}$$

By Completeness Theorem,

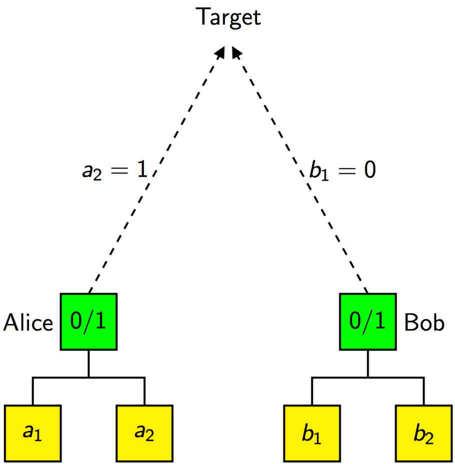
$$\sum_{j<k} \phi_j \leq k - 1$$

is provable in PTL.

Abramsky and Hardy[1] call inequalities of this form *logical Bell's inequalities*.

[1] Samson Abramsky and Lucien Hardy. Logical Bell Inequalities. Physical Review A, 85(062114):1–11, 2012.

Alice/Bob



Observing entangled particles

$$
\begin{aligned}
p_0 &= \text{“Alice measurement at } 0^\circ \text{ has outcome } \uparrow\text{.”},\\
p_1 &= \text{“Bob measurement at } 180^\circ \text{ has outcome } \uparrow\text{.”},\\
p_2 &= \text{“Alice measurement at } 60^\circ \text{ has outcome } \uparrow\text{.”},\\
p_3 &= \text{“Bob measurement at } 120^\circ \text{ has outcome } \uparrow\text{.”},
\end{aligned}
$$

We measure four multi-teams, corresponding to four settings of directions of measuring the spin. Probability table:

	$(1,1)$	$(0,1)$	$(1,0)$	$(0,0)$
(p_0, p_1)	$1/2$	0	0	$1/2$
(p_0, p_3)	$3/8$	$1/8$	$1/8$	$3/8$
(p_1, p_2)	$3/8$	$1/8$	$1/8$	$3/8$
(p_2, p_3)	$1/8$	$3/8$	$3/8$	$1/8$

Consider now the following propositional formulas:

$$\begin{aligned}\phi_0 &= (p_0 \wedge p_1) \vee (\neg p_0 \wedge \neg p_1)\\ \phi_1 &= (p_0 \wedge p_3) \vee (\neg p_0 \wedge \neg p_3)\\ \phi_2 &= (p_1 \wedge p_2) \vee (\neg p_1 \wedge \neg p_2)\\ \phi_3 &= (\neg p_2 \wedge p_3) \vee (p_2 \wedge \neg p_3)\end{aligned}$$

If there was a multi-team X with the above four as projections, then $[\phi_0]_X = 1$ and $[\phi_j]_X = \frac{6}{8}$ for $j = 1, 2, 3$. Furthermore, the formula $\bigwedge_{j<4} \phi_j$ is clearly contradictory. So then by (2) we must have that

$$\sum_{j<4} [\phi_j]_X = 1 + 3 \cdot \frac{6}{8} = 3 + \frac{1}{4} \leq 3,$$

a contradiction.

Conclusion

- The above four multi-teams cannot arise from one single multi-team.
- Abramsky-Gottlob-Kolaitis: This is an instance of the universal database problem (NP-complete).
- Abramsky: No “global section”.
- Existence of global section is an example of a sentence of dependence logic.

The multi-team approach is not the right approach!

- For ϕ_0, ϕ_1, ϕ_2 and ϕ_3 as above:

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 \leq 3 \qquad (3)$$

  is provable in PTL.
- But our experiment (and QM) supports

$$\phi_0 + \phi_1 + \phi_2 + \phi_3 > 3.$$

  Thus PTL is not the right "quantum" logic.
- Therefore, the multi-team (probabilistic team) approach is not the right approach to describing the logic of quantum phenomena.

Quantum teams

- Suppose $(Q_i)_{i \in \Omega}$ is a finite sequence of finite non-empty sets of proposition symbols.
- A *quantum team* on $(Q_i)_{i \in \Omega}$ is an indexed set X of valuations $v(i)$ such that $v(i)$ is a truth-value assignment to the proposition symbols in Q_i for $i \in \Omega$.
- We call $\{Q_i : i \in \Omega\}$ the *support* of X and denote it $\mathrm{Sp}(X)$.
- The set $\bigcup_{i \in \Omega} Q_i$ is called the *domain* of X and denoted $dom(X)$.

	p_0	p_1	p_2	p_3
0	1	1	—	—
1	1	1	—	—
2	1	1	—	—
3	1	1	—	—
4	0	0	—	—
5	0	0	—	—
6	0	0	—	—
7	0	0	—	—
8	1	—	—	1
9	1	—	—	1
10	1	—	—	1
11	0	—	—	1
12	1	—	—	0
13	0	—	—	0
14	0	—	—	0
15	0	—	—	0
16	—	1	1	—
17	—	1	1	—
18	—	1	1	—
19	—	0	1	—
20	—	1	0	—
21	—	0	0	—
22	—	0	0	—
23	—	0	0	—
24	—	—	1	1
25	—	—	1	0
26	—	—	1	0
27	—	—	1	0
28	—	—	0	1
29	—	—	0	1
30	—	—	0	1
31	—	—	0	0

- Given a finite set U of proposition symbols and a quantum team X on $(Q_i)_{i \in \Omega}$, we let $\Omega_U = \{i \in \Omega \mid U \subseteq Q_i\}$. We use this notation only if $\Omega_U \neq \emptyset$.
- We can define a new quantum team X_U by letting $v_U(i) = v(i) \upharpoonright U$ for $i \in \Omega_U$.
- For each valuation w on U we define

$$P_X^U(w) = \frac{|\{i \in \Omega_U : v_U(i) = w\}|}{|\Omega_U|}.$$

- This extends canonically to a definition of the probability of a propositional formula ϕ with its proposition symbols in U such that $\Omega_U \neq \emptyset$:

$$[\phi]_X^U = P_X^U(\{i \in \Omega_U \mid v_U(i)(\phi) = 1\}).$$

For the above quantum team, formulas ϕ_j, and an obvious choice of $(V_j)_{j<4}$

$$\sum_{j<4}[\phi_j]_X^{V_j} = 3 + \frac{1}{4}.$$

Hence the “false” Bell's Inequality is not true in this quantum team.

A logic for arguing about probabilities in quantum teams

- We build a formal language for reasoning about statements of the form "the probability of ϕ, with propositional variables in V, is at least 0.125".
- We shorten this to $\phi^V \geq 0.125$.
- We allow more general statement $a_0\phi_0^{V_0} + \ldots + a_{k-1}\phi_{k-1}^{V_{k-1}} \geq c$.

Quantum team logic

Suppose $(\phi_j)_{j \leq k}$ are propositional formulas, $(a_j)_{j<k} \in \mathbb{Z}^k$, $c \in \mathbb{Z}$ and $(V_j)_{j<k}$ a sequence of finite sets of proposition symbols, so that the proposition symbols of ϕ_j are in V_j for every $j < k$. Then

$$a_0 \phi_0^{V_0} + \ldots + a_{k-1} \phi_{k-1}^{V_{k-1}} \geq c$$

is an atomic formula of QTL.
The set of formulas of QTL is defined as follows:

- atomic formulas are formulas;
- if α is a formula, then $\neg\alpha$ is a formula;
- if α and β are formulas, then $\alpha \wedge \beta$ is a formula.

Semantics

Let α be a formula of QTL and X a quantum team such that every superscript set V occurring in α is included in some set in the support of X. We define by induction on α the relation "X satisfies α", $X \models \alpha$, in the following way:

- $X \models \sum_{j<k} a_j \phi_j^{V_j} \geq c$ iff $\sum_{j<k} a_j [\phi_j]_X^{V_j} \geq c$;
- $X \models \neg\alpha$ iff $X \not\models \alpha$;
- $X \models \alpha \wedge \beta$ iff $X \models \alpha$ and $X \models \beta$.

The deductive system of QTL

Below $\mathcal{V}$ is a finite set of finite sets of proposition symbols. We apply the below rules only when every superscript set in any of the formulas is included in some set in $\mathcal{V}$.

Propositional reasoning

A) $\vdash_{\mathcal{V}} \alpha$, for α a propositional tautology.

B) If $\vdash_{\mathcal{V}} \alpha \to \beta$ and $\vdash_{\mathcal{V}} \alpha$, then $\vdash_{\mathcal{V}} \beta$ (modus ponens).

Probabilistic reasoning

C) $\vdash_{\mathcal{V}} \phi^{V} \geq 0$.

D) $\vdash_{\mathcal{V}} (\phi \vee \neg\phi)^{V} = 1$.

E) $\vdash_{\mathcal{V}} (\phi \wedge \psi + \phi \wedge \neg\psi)^{V} = \phi^{V}$ (additivity).

F) If $\phi \equiv \psi$ in propositional logic, then $\vdash_{\mathcal{V}} \phi^{V} = \psi^{V}$.

Linear inequalities

G) $\vdash_{\mathcal{V}} \phi^V \geq \phi^V$.

H) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \phi_j^{V_j} \geq c \Leftrightarrow \vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} + 0\psi^V \geq c$.

I) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \geq c \Leftrightarrow \vdash_{\mathcal{V}} \sum_{j<k} a_{\sigma(j)} (\phi_{\sigma(j)}^{V_{\sigma(j)}} \geq c$ (for σ permutation on k).

J) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \geq c \wedge \sum_{j<k} b_j \phi_j^{V_j} \geq d \Leftrightarrow \vdash_{\mathcal{V}} \sum_{j<k} (a_j + b_j) \phi_j^{V_j} \geq c + d$.

K) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \geq c \Leftrightarrow \vdash_{\mathcal{V}} \sum_{j<k} d a_j \phi_j^{V_j} \geq c$ (for $d > 0$).

L) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \geq c \vee \sum_{j<k} a_j \phi_j^{V_j} \leq c$.

M) $\vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} \geq c \Rightarrow \vdash_{\mathcal{V}} \sum_{j<k} a_j \phi_j^{V_j} > d$ (for $c > d$).

Change of support

N) $\vdash_{\mathcal{V}} \phi^V = 0$ and $V \subseteq V' \in \mathcal{V} \Rightarrow \vdash_{\mathcal{V}} \phi^{V'} = 0$.

O) $\vdash_{\mathcal{V}} \phi^V = 1$ and $V \subseteq V' \in \mathcal{V} \Rightarrow \vdash_{\mathcal{V}} \phi^{V'} = 1$.

The Main Result

Theorem (Completeness)

Let ϕ be a formula of QTL. Then ϕ is provable in QTL if and only if it is true in every quantum (multi i.e. probabilistic) team.

Hence the "false" Bell's Inequalities are not provable in QTL.

Summary

- Multi-teams model probabilistic propositional logic.
- Validities in multi-teams can be completely axiomatized.
- From the point of view of quantum phenomena this approach is not satisfactory, as “false” Bell’s Inequalities are provable.
- Quantum teams generalise multi-teams by allowing the circumstance that some attributes cannot be simultaneously measured.
- Validities in quantum teams can be completely axiomatised.
- “False” Bell’s Inequalities are not provable.
- Is this the right “Quantum Logic”?

Question

- Can we find axioms such that ϕ is provable if and only if ϕ is true in every real quantum team, i.e. a quantum team that can be physically realised by an experiment (or by QM)?

Thank you!