Magnetic Laplacian

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Magnetic Laplacian

- Talk I: A general Panorama
- Talk II: "Can we hear the magnetic locus of the magnetic field?"

Some References

- N.Raymond Lecture on the Magnetic Laplacian (Tunis workshop 2012)
- J.Bellissard, Quantum Hall effect and non-commutative geometry
- (- , Germinet.F, Raikov.G) Quantization of the edge current along a magnetic barrier.
- (- , Hislop.P, Soccorci.E) Transport analysis of magnetic edge-currents (*work in progress*).
- (- , Popoff,N) Spectral asymptotics and numerical simulations for magnetic wave-guide (*work in progress*).
- (- , Raymond.N) Semiclassical analysis with vanishing magnetic fields.

Laplacian

- M be a compact oriented manifold (dimension $n \geq 2$);
- g a Riemannian metric,
- $\mathbf{d}$ the usual exterior derivative.

Usual Laplacian

$$\Delta := \mathbf{d}^*\mathbf{d}$$

acting on $C_c^\infty(M)$.
In local coordinate, (Einstein convention)

$$\Delta = \sqrt{|g|}^{-1}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right)$$

First Definition of the Magnetic Laplacian

- M be a compact oriented manifold (dimension $n \geq 2$).
- g a Riemannian metric
- B a closed real-valued 2-form
- Assume B is exact and choose A a 1-form such that $dA = B$.

The Magnetic Laplacian

Let $\mathbf{d}_A := \mathbf{d} - iA$, then

$$\Delta_A := \mathbf{d}_A^* \mathbf{d}_A$$

acting on $C_c^\infty(M)$

Semi-Classical Magnetic Laplacian

Since now we are looking for $n = 2$ and Euclidian setting.

Magnetic Laplacian

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2, \quad \mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2) \ ,$$

- β magnetic field such that

$$\beta = \nabla \times \mathbf{A} = \partial_{x_1}\mathbf{A}_2 - \partial_{x_2}\mathbf{A}_1$$

  magnetic field generated by $\mathbf{A}$.
- h the Semi-classical parameter (Planck constant).

Very important feature

Gauge invariance

- ϕ a smooth real-valued function.

$$e^{-i\phi}\left(-i\nabla + \mathbf{A}\right)e^{i\phi} = \left(-i\nabla + \mathbf{A} + \nabla\phi\right)$$

Thus by unitary

$$e^{-i\phi}\mathcal{L}_{h,\mathbf{A}}e^{i\phi} = \mathcal{L}_{h,\mathbf{A}+\nabla\phi}$$

and finally we have that

$\Rightarrow$ The spectrum of $\mathcal{L}_{h,\mathbf{A}}$ depends only on β and not of the choice of the potential.
(This is the the magnetic field that is the effective physical object)

First Reference Operator

Quantum Harmonic Oscillator

The most fundamental operator of quantum mechanic:

$$h_0 := P^2 + Q^2 \text{ acting on } L^2(\mathbb{R})$$

with $\{P, Q\}$ conjugate operators of momentum and position.

$$(P\varphi)(x) = -i\hbar\partial_x\varphi(x)$$

$$(Q\varphi)(x) = \mathbf{x}\varphi(x)$$

Commutation relation

$$[P, Q] = i\hbar$$

Eigenpairs

- The eigenvalues are given by:

$$\sigma(h_0) = \{2n+1, n \in \mathbb{N}^*\}$$

- The eigenfunction are given by:

$$\varphi_n = c_n exp\left(\frac{-x^2}{2}\right) H_n(x)$$

where H_n are the the so-called Hermite polynomials and c_n are normalization constants.

Applications

- Geometry :
  - $\mathcal{L}_{h,\mathbf{A}}$: Covariant Laplacian w.r.t a connection whose curvature is β.
    (Relation between semi-classical spectral asymptotics of magnetic Laplacians and SubRiemannian geodesics, see Montgomery.)
- Physic :
  - 2D quantum particle in a magnetic field β.
  - Supra-conductivity: third critical field of the Ginzburg-Landau fonctional,
  - Quantum Hall effect: topologically quantized transport.

Magnetic Reference Operator (Central role !!!)

Landau Operator on $\mathbb{R}^2$

- $B(x) \equiv B > 0$: constant magnetic field
- We choose the so-called symmetric gauge given by $A_0(x,y) = (0, Bx)$ generating B.

We can then write

$$H_L = H(A_0) = D_x^2 + (D_y - Bx)^2\,,\; D_{x,y} = -i\partial_{x,y}$$

Landau Hamiltonian

Magnetic Field breaks the Symmetries

Symmetries

$$H_L = P_1^2 + P_2^2$$

with

$$i[P_1, P_2] = B \neq 0$$

(Same structure than the Q-harmonic oscillator)

Consequences

Magnetic translations not commute and form a *projective representation*: Let $\gamma \in \mathbb{R}^2$,

- Second direction symmetry:

  $$(T(\gamma_2)\varphi)(x, y) := \varphi(x, y - \gamma_2)$$

  (usual translation operator)

- First direction symmetry:

  $$(U(\gamma_1)\varphi)(x, y) := e^{-iB\gamma_1 y}\varphi(x - \gamma_1, y)$$

  (called magnetic translations (also called Zak translation) due to gauge invariance)

Consequence on the algebraic structure

- Let $\mathcal{Z}$ a locally compact group
- The symmetries form a ζ-projective unitary representation of $\mathcal{Z}$ on $\mathcal{H} = L^2(\mathbb{R})$, $\{U_x\}_{x\in\mathcal{Z}}$ i.e

$$U_x U_y = \zeta(x,y) U_{x+y} \textit{ pour tout } x, y \in \mathcal{Z}$$
$$U_x^* = \overline{\zeta}(x,-x) U_{-x} \text{ for all } x \in \mathcal{Z}.$$

with ζ a 2-cocycle.

Then the observables are the bounded operators having the same symmetries also called covariant: operator A such that

$$U_x A U_x^* = A \Leftrightarrow A \in \{U_x\}'_{x\in\mathcal{Z}}$$

Let $\mathcal{K}_\infty$ the Von-Neumann algebra of such observables, Then $\mathcal{K}_\infty$ is isomorphic to a twisted-cross product
$\Rightarrow$ the K-theory is not trivial (Core of the Quantum Hall effect)

Spectrum of the Landau Operator

The Landau operator as an infinite sum of Q-harmonic oscillators

Partial Diagonalization

- Let $\mathcal{F}_y$ the usual Fourier transform w.r.t the second direction.
- Let $\hat{\mathcal{F}} = id_x \otimes \mathcal{F}_y$

Provides the unitary equivalence

$$H_L \simeq \int_{\mathbb{R}}^{\oplus} \mathfrak{h}_L(\tau) d\tau$$

with

$$\mathfrak{h}(\tau) = D_x^2 + (\tau - Bx)^2 \quad \text{on } L^2(\mathbb{R}) \text{ with } \tau \in \mathbb{R}$$

After a rescaling we get that

$$\mathfrak{h}(\tau) \simeq B\left(D_x^2 + x^2\right) \quad \text{on } L^2(\mathbb{R}) \text{ for any } \tau \in \mathbb{R}$$

Then

$$\sigma(H_L) = \{B(2n+1),\ n \in \mathbb{N}^*\} \quad (\text{The so-called Landau levels })$$

But this time each eigenvalues is infinitely degenerated !

Trivial transport

H_L is " *spectrally flat* " : for any $k \in \mathbb{N}^*$ and $\mu_k(\tau) \in \sigma(\mathfrak{h}_L(\tau))$,

$$\partial_\tau \mu_k(\tau) \equiv 0$$

.

Hall Effect

hall.png

- Let N la density of electrons and e the elementary charge. equilibrium equation:

$$Ne\mathcal{E} + \mathbf{j} \wedge B = 0 \ .$$

- Solving this equation w.r.t $\mathbf{j}$:

$$\mathbf{j} = \frac{Ne}{|B|^2} B \wedge \mathcal{E} = \sigma \mathcal{E}$$

where σ is *the tensor of conductivity* .
(Only the off-diagonal component are non-zero)

Quantum Hall Effect

- One century later (1981), von-Klitzing discovered the Quantum conterpart: Temperature ~ 0 making the Quantum phenomenon predominant. (Nobel Prize and second one for Laughlin 10 years after)
- Particular features of the QHE:
  1. Occurrence of Plateaux
  2. Quantization of the conductance
  3. Stability under pertubations (Topological nature)

plateauxhall.pdf

Introduction to the mathematical theory

- Dynamic: Schrödinger equation

$$i\partial_t \varphi_t = H_\omega \varphi_t$$

with H_ω the ergodic random magnetic Hamiltonian

$$H(\mathcal{A}, V_\omega) = (-i\nabla - \mathcal{A}(x))^2 + V_\omega(x) \text{ acting on } \mathcal{H} := L^2(\mathbb{R}^2)\ ,$$

with $\mathcal{A}$ the magnetic potential, V_ω the scalar potential associated to the media and ω running over a probability space $(\Omega, \mathbb{P})$.

- We compute the conductance via the so-called response linear theory: *the conductance can be defined as the first order response coefficient of the system under an external perturbation (Electrical field)*:
We look at a perturbation (adiabatic) of the system:

$$H_\omega(t) = H_\omega + \mathbf{E}(x, t, \eta)$$

Mathematical Definition of the Hall Conductance

Let us fix an instant t_0,

Observable Current

- The observable Current (also called velocity) :

$$\mathcal{J}(t_0) = \dot{\mathbf{X}}(t_0)$$

  where by definition of the quantum dynamic we have:

$$\dot{\mathbf{X}}(t_0) := i[\mathbf{x}, H_{t_0}]$$

Conductance

- Let $\rho^{E_F}_{f.d,t_0}$ the matrix of density which represents the state of the system at time t_0.
  The Quantum statistical mean of the observable current is by definition

$$\mathbf{J}(E, \eta, t_0, E_F) \ = \mathcal{T}(\rho^{E_F}_{f.d,t_0} \dot{\mathbf{X}}(t_0))$$

  $\mathcal{T}$ is a natural trace associated to the algebra of observable (namely the trace per unit volume).
- Then the the Hall conductance via the linear response theory is defined as

$$\sigma_H(H) := \lim_{\eta \to 0} \partial_E \mathbf{J}(E, \eta, t_0, E_F)_{|\mathbf{E}=0}$$

One of the most beautiful Result of Mathematical Physic!:

The Connes-Bellissard Theorem

As explained later, The spectrum of the Landau operator, H_L is given by $\sigma(H_L) = \{B(2n+1),\ n \in \mathbb{N}^*\}$, the Landau level.

Let $\mathbb{G}_n$ the n-th gap between two level.

If we take the zero temperature then $\rho_{f.d,t_0}^{E_F}$ becomes an orthogonal projection called the Fermi-Dirac projection denoted $\mathbf{P}_{E_F}$.

Connes-Bellissard Theorem

If $E_F \in \mathbb{G}_n$
Then

$$\sigma_{i,j}^{E_F}(H_L) = \mathcal{T}(\mathbf{P}_{E_F}[\partial_i \mathbf{P}_{E_F}, \partial_j \mathbf{P}_{E_F}]) = Ch(\mathbf{P}_{E_F}) = \mathbf{n}$$

with ∂_i a $*$-derivation.

This result can be view as a non-commutative Gauss-Bonnet theorem.

Dual system

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Schulz-Baldes-Kellendonk Theorem, a duality theorem

- Let $\sigma_b^{E_F}(H_L)$ the Hall conductance previously defined.
- Let H_L^+ the Landau operator restricted to an half-plane with adapted boundary condition.
- Let $\sigma_e^I(H_L^+)$ the edge-conductance which is the analogue of the previous one but for edge-system.

Bulk-Edge correspondence

Let $E_F \in \mathbb{G}_n$ and $I \subset \mathbb{G}_n$.
Then we have

$$\sigma_b^{E_F}(H_L) = \sigma_e^I(H_L^+) \quad (= \mathbf{n})$$

In some sense we can view this theorem as a non-commutative Stockes theorem.

Topologically Quantized Transport

Two ways to create Topologically quantized Transport :

Bulk-systeme

Adiabatic Perturbation

$\sim$

Edge-system

Perturbation w.r.t the partial symmetry

$\Downarrow$

Usual edge-model

1. $H = H_L^+$: Geometric confining
2. $H = H_L + V_0$: Scalar confining

- H_L^+ Landau operator restricted to the half-plane $\mathbb{R}^+ \times \mathbb{R}$ with Dirichlet b.c
- V_0 translation-invariant scalar potential confining to the half-plane $\mathbb{R}^+ \times \mathbb{R}$.

Magnetic confining

Generalized Iwatsuka Operator

- Idea: *consider a translation-invariant magnetic field*
- ex:

$$H = D_x^2 + (D_y - \mathcal{A}_{GI}(x))^2, \quad \mathcal{A}(x) \in C^\infty(\mathbb{R})$$

$$\mathcal{A}_{GI}(x) = \int_0^x \beta(s)ds$$

β s.t

$$\lim_{s \to \pm\infty} \beta(s) = B_\pm$$

and monotone.

snakestate.pdf

Topological magnetic transport

Transport

- Symmetries $\Rightarrow$ Diagonalisation

$$H \cong \int_{\mathbb{R}}^{\oplus} \mathfrak{h}(\tau) d\tau, \quad \mathfrak{h}(\tau) = D_x^2 + (\tau - \beta(x))^2$$

- $\forall n \in \mathbb{N}$ and $\mu_n(\tau) \in \sigma(\mathfrak{h}(\tau))$, we have that

$$|\partial_\tau \mu_n(\tau)| > 0 \Rightarrow \text{ Transport}$$

edge_current2.png

Results : *Topological Quantization of the conductance*

- Let $\sigma^I(H)$ the edge-condctance and I a real interval included in the n-th spectral gap of H_L

$$\sigma^I(H) = \begin{cases} n & \text{if } sign(B_-)\cdot sign(B_+) = 1 \ (usual) \\ 2n & \text{if } sign(B_-)\cdot sign(B_+) = -1 \ \text{new!} \end{cases}$$

- Stability of the quantization:

$$\sigma(H(A + a_\omega)) = \sigma(H(A))$$

with $a_\omega \sim O(B^{\frac{1}{2}})$ supported on the half-plane.

	Topological quantization	Semi-classical spectral asymptotics for a well-problem
Geometric	n	Solved
Electric	n	Solved
Magnetic sign-definite	n	Solved
Magnetic non sign-definite	2n	Open(partially)

Analysis of a generic model of magnetic wave-guide

Model operator

$$H_s = D_x^2 + (D_y + |x|)^2 \quad \text{on } L^2(\mathbb{R}^2)$$
$$\simeq \int_{\mathbb{R}}^{\oplus} \mathfrak{h}_{sym}(\tau) d\tau$$

with

$$\mathfrak{h}_{sym}(\tau) = D_x^2 + (\tau + |x|)^2 \ \text{ on } L^2(\mathbb{R}) \ .$$

Vtaunegatif.pdf

Figure : Potential of $\mathfrak{h}_{\mathrm{sym}}(\tau)$ for $\tau < 0$, here for $\tau = -2$.

Vtaupositif.pdf

Figure : Potential of $\mathfrak{h}_{\mathrm{sym}}(\tau)$ for $\tau > 0$, here for $\tau = 4$.

Symmetrization

$$\mathfrak{h}_{sym}(\tau) \simeq h_N(\tau) \oplus h_D(\tau)$$

- with
$$h_{N,D}(\tau) = -\partial_x^2 + (x-\tau)^2 \ \text{ sur } L^2([0,+\infty))$$

Spectral Asymptotics

Let $\mu_n^{N,D}(\tau)$ the n-th eigenvalues of the operator $h_{N,D}(\tau)$.

Spectral Asymptotiques at $\tau \to -\infty$

- $\mu_n^N(\tau) = \tau^2 - 2^{2/3}\mathcal{Z}'_n|\tau|^{2/3} + o\left(|\tau|^{2/3}\right)$
- $\mu_n^D(\tau) = \tau^2 - 2^{2/3}\mathcal{Z}_n|\tau|^{2/3} + o\left(|\tau|^{2/3}\right)$.
  with $\mathcal{Z}_n$ (respc. $\mathcal{Z}'{}_n$) the zeros of the Airy function (resp. of the derivative of the Airy function).

Spectral Asymptotics at $\tau \to +\infty$

- $\mu_n^D(\tau) = 2n - 1 + \frac{2^n}{(n-1)!\sqrt{\pi}}\tau^{2n-1}e^{-\tau^2}\left(1 - \frac{n^2-n+1}{2\tau^2} + O\left(\frac{1}{\tau^4}\right)\right)$
- $\mu_k^N(\tau) = 2n - 1 - \frac{2^n}{(n-1)!\sqrt{\pi}}\tau^{2n-1}e^{-\tau^2}\left(1 - \frac{n^2-n-1}{2\tau^2} + \mathcal{O}\left(\frac{1}{\tau^4}\right)\right)$.

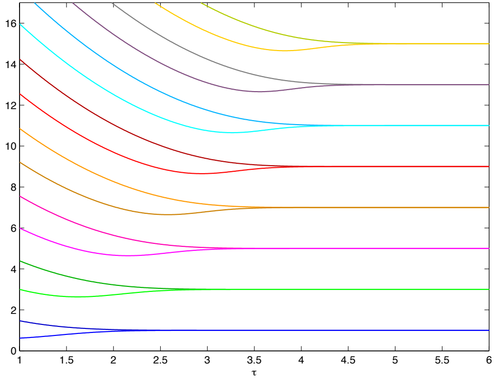

Figure : Eigenvalues $\mu_n^D(\tau)$ and $\mu_n^N(\tau)$ with $\tau \in [1, 6]$. Done for us by Nicolas Popoff.

Key fact: Occurrence of sign-definite non degenerate positive minimum at the bottom of each band

Approximation theory

- Question: What is the naturally well-adapted approximation theory?
- Answer from Physics of solids: The Effective mass Theory (EMT)

Idea of the effective mass theorem

1. H_0 a 2D fibred Hamiltonian with

$$H_0 \simeq \int_{\mathbb{R}_x}^{\oplus} \mathfrak{h}(\tau) d\tau$$

2. $V(y)$: Smooth and localized perturbation depending only of the second direction

Assumption: The bottom of the spectrum is given by a sign-definite positive non-degenerate minimum at τ_0.

Effective Hamiltonian for $H = H_0 + V$?

$$\mu_0(\tau) \sim \mu(\tau_0) + \mu_0''(\tau_0)\frac{\tau^2}{2}. \text{ for } \tau \sim \tau_0$$

$$H_{EM} - \mu_0(\tau_0) := \frac{\mu_0''}{2}(\tau_0) D_y^2 + V(y)$$

Mathematical signature of this magnetic phenomenon

Finally, we get the following mathematical signatures of the phenomenon:

1. Sensitivity to the gradient of the field and not to the field it-self.
2. Change of the magnetic field's sign
3. Occurrence of sign-definite positive minimum at the bottom of the band-spectrum: EMT approximation.

Semi-classical and microlocal analysis of the vanishing magnetic field phenomenon

R.Montgomery:" *Can we hear the zero locus of a magnetic field?"* '94

Magnetic Laplacian

$$\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2, \quad \mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2) \ ,$$

$\beta = \nabla \times \mathbf{A}$ the magnetic field generated by $\mathbf{A}$.

Non sign-definite Magnetic Well

1. $\beta(x) \underset{|x|\to+\infty}{\to} +\infty$,
2. β vanishes along $\Gamma = \{\gamma(s), s \in \mathbb{R}\}$, a smooth closed curve.
3. β is non-positive inside of Γ and non-negative outside.

Problem

Compute the semi-classical spectral asymptotics

$$\lim_{h\to 0} \lambda_n(h)$$

with $\lambda_n(h) \in \sigma(\mathcal{L}_{h,\mathbf{A}})$ and $n \in \mathbb{N}$.

Coordinate near the zero locus curve

- standard tubular coordinates :

$$\Phi(s,t) = \gamma(s) + t\nu(s),\ (s,t) \text{ near } \Gamma$$

  where

  - $\nu(s)$: the interior normal unit vector to Γ
  - $k(s)$: the curvature.
- Let $\tilde{\beta}(s,t) = \beta(\Phi(s,t))$ such that: $\tilde{\beta}(s,0) = 0$.

- The derivative with respect to the normal of β on Γ :

$$\delta : s \mapsto \partial_t \tilde{\beta}(s,0)\ .$$

Main assumption

δ admits a unique non-degenerate and positive minimum at the point x_0.

Let us note $\delta_0 = \delta(0)$ and assume that $x_0 = (0,0)$.

Model Operator and homogenized operator

Model Operator : Anharmonic oscillator

Montgommery Operator:

$$H_{\eta,\delta} = D_t^2 + (-\eta + \frac{\delta}{2}t^2)^2 \ ,$$

$\eta \in \mathbb{R}$ and $\delta > 0$.

Known Results : Let $\nu_1(\eta)$ the first eigenvalue and u_η the associated eigenfunction.

Then

1. $\eta \mapsto \nu_1(\eta)$ admits a unique minimum non-degenerate at point η_0.
2. $\nu''(\eta_0) > 0$

Key idea

In order to build a multi-scale expansion we need a second well-choosen operator acting in the second direction (EMT): *Homogenized Operator* :

$$\mathcal{H}_\alpha := \frac{\nu''(\eta_0)}{2} D_s^2 + \alpha s^2$$

$\alpha \in \mathbb{R}$.

Past results

1. (Montgomery '94):

$$\lim_{h\to 0} \frac{\lambda_1(h)}{h^{\frac{2}{3}}} = \nu_1(\eta_0)$$

2. (Helffer-Kordyukov, '09):

if $C^{-1}d(x,\Gamma)^k \leq \beta(x) \leq Cd(x,\Gamma)^k$, for $k \in \mathbb{N}^*$, then

$$\lim_{h\to 0} h^{-\frac{2k+2}{k+2}} \lambda_1(h) = \nu_1(\eta_0)\delta_0^{\frac{2}{k+2}}$$

Theorem

Theorem

Under the previous assumption, for all $n \geq 1$, there exist a sequence $(\theta_j^n)_{j\geq 0}$ and $h_0 > 0$ such that for $h \in (0, h_0)$, we have :

$$\lambda_n(h) \sim h^{4/3} \sum_{j\geq 0} \theta_j^n h^{j/6}$$

where :

$$\theta_0^n = \delta_0^{2/3} \nu_1(\eta_0), \quad \theta_1^n = 0,$$

$$\theta_2^n = \delta_0^{2/3} C_0 + \delta_0^{2/3}(2n-1)\left(\frac{\alpha\nu(\eta_0)\nu''(\eta_0)}{3}\right)^{1/2}$$

α et C_0 constants depending of $\delta_0, \delta''(0)$ et $k(0)$.

- $\nu(\eta_0) \sim 0.57, \quad \eta_0 \sim 0.35$ [BN]

Thank you for your attention!!