Fractal dimensions of multifractals with applications

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Inertial particles in turbulence

Small particles in turbulent flow u(t, x) obey

$$\dot{\boldsymbol{v}} = -\frac{\boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{x})}{\tau}$$

at small dimensionless relaxation time St (Maxey, 1987):

$$\dot{\boldsymbol{v}} = \boldsymbol{u} - \tau\left(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right)$$

particles accumulate on evolving multifractal boundary between vortices (Falkovich, Stepanov, IF, 2002):

$$\nabla \cdot \boldsymbol{v} = \tau\left(\omega^2/2 - s^2\right)$$

Compressibility is small - pair correlation of concentration:

$$\langle n(0)n(\boldsymbol{r})\rangle = \langle n\rangle^2 \left(\frac{\eta}{r}\right)^{2C_{KY}}$$

C_{KY} is the Kaplan-Yorke codimension. Collide more often.

Multifractality holds up to St~1 (Bec, 2003): sling effect(FSI)

Droplets in warm clouds

In clouds ratio Fr of turbulent acceleration to g is small (0.5 for strong turbulence with dissipation rate of 2000 cm^2/s^3): sedimentation is not negligible.

Gravity has soothing action on spatial motion (Falkovich Pumir 2007), induces multifracrality at St>>1 (Bec et al 2014).

Theory at Fr $\rightarrow$ 0: IF, Park, Harduf, Lee 2015:

Flow of droplets can be defined at however large St. Particles separate mainly horizontally so that vertical configurations are quasi-stable: columns form in space. Lyapunov exponent and pair correlations are given via energy spectrum E(k):

$$\lambda_1 \tau = \frac{\pi \int_0^\infty E(k) k dk}{8g} \qquad D_{KY} = \frac{3\pi \int_0^\infty E(k) k dk}{4g} \qquad \langle n(0) n(\boldsymbol{r}) \rangle = \langle n \rangle^2 \left(\frac{\eta}{r} \right)^{2D_{KY}}$$

confirmed by DNS of droplets' motion in the Navier-Stokes flow.

Flow description

Flow of droplets in warm clouds obeys

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{v}, \quad \frac{d\boldsymbol{v}}{dt} = -\frac{\boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{x}(t))}{\tau} + \boldsymbol{g}$$

In contrast with Maxey formula we cannot derive the velocity of droplet via local flow and its derivatives: the relation is non-local in space and time. Implicit proof:

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{\boldsymbol{v} - \boldsymbol{u}}{\tau} + \boldsymbol{g}$$

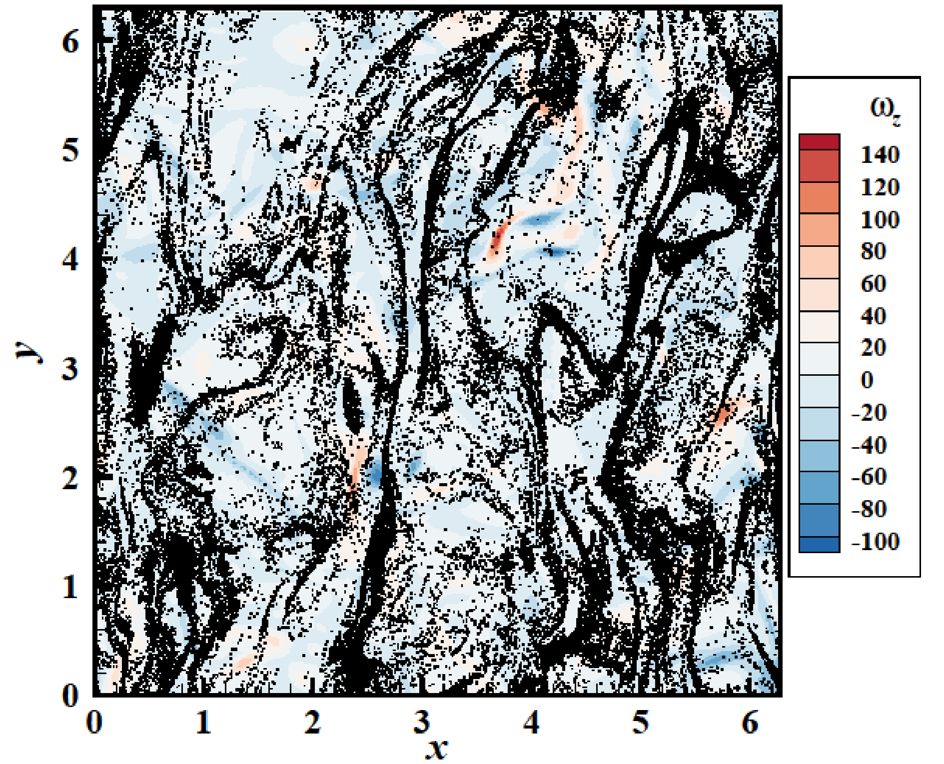
This is single valued if $\sigma_{ik}(t) \equiv \nabla_k v_i(t, \boldsymbol{x}(t))$ do not blow up,

$$\dot{\sigma} + \sigma^2 = -\frac{\sigma - s}{\tau}, \qquad \sigma_l \equiv \int_{-\infty}^{t} dt' \exp\left(-\frac{t - t'}{\tau}\right) s(t') \frac{dt'}{\tau}$$

which is true at Fr<<1. Droplets pass many vortices in their reaction time τ which smoothens the sling effect.

New starting point for theory of droplets motion in warm clouds

Distribution at Fr<<1 is multifractal, driven by weakly compressible strongly anisotropic flow. Clustering is uncorrelated with vorticity, theory is possible due to compressibility's weakness and small correlation time. Our proposal: derive theory a Fr<<1, interpolate to higher Fr (where multifrality continues to hold Bec et al. 2014)



Theory of collisions of dropletsv in warm clouds

Collision kernel is derived in S. Lee, C. Lee, IF 2019. It is given by the product of average velocity difference of colliding droplets, found using Gaussianity of flow gradients and radial distribution function or pair correlation function of concentration that is angle-dependent, peaked on vertical

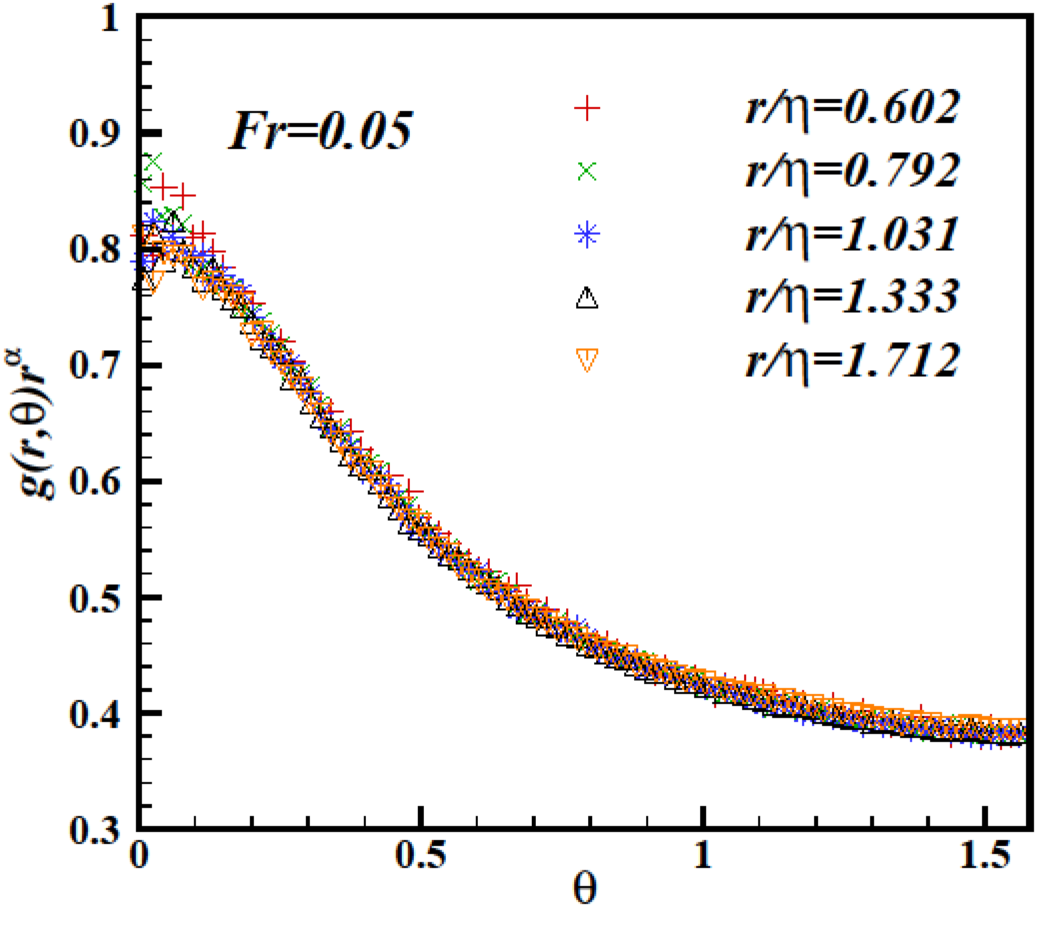
$$g(r,\theta)=\left(\frac{l_c}{r\sin\theta}\right)^{\alpha},\quad r<l_c,\quad \theta>\theta^*;\quad g(r,\theta)=\left(\frac{cg\tau^2}{r}\right)^{\alpha},\quad r<g\tau^2,\quad \theta<\theta^*$$

The theory holds up to parameters holding in precipitating clouds Fr=0.05 where the spectrum formula fails however correlation codimension is twice the KY codimension

$$\alpha=2\left|\frac{\sum_{i=1}^{3}\lambda_i}{\lambda_3}\right|=\frac{3\pi\int_0^\infty E(k)kdk}{2g}=12c_0Fr$$

Separable pair correlations

The data collapses demonstrating separability of the RDF in



Applications

Large, growing number of problems can be solved using simple recipe: find limit where the particles form flow, write down multifractal statistics. The flow is usually weakly compressible so general theory of multifractals formed by transport by weakly compressible flows (IF 2012) applies. Interpolate beyond the region of validity. This works well for pair correlations however for collision kernel conclusions from the limit can fail soon because of the sling effect.

Maxey-type formulas: phytoplankton cells, phoretic particles, bubbles

Maxey-type formulas

Fine bubbles (with Shim, S. Lee, C. Lee 2018)

$$\dot{\boldsymbol{v}} = 3\left[\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right] - \frac{\boldsymbol{v}-\boldsymbol{u}-\boldsymbol{v}_g}{\tau} + \boldsymbol{\omega}\times(\boldsymbol{v}-\boldsymbol{u})$$

Reduction in the whole range of equation's validity

$$\boldsymbol{v} = \boldsymbol{u} + (\tau v_g \omega_y, -\tau v_g \omega_x, v_g)$$

Coincides with reduction for phytoplankton cells

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{u}(t,\boldsymbol{x}(t)) + \phi\hat{\mathbf{p}}(t); \quad \frac{d\hat{\mathbf{p}}}{dt} = \frac{\hat{\boldsymbol{k}} - (\hat{\boldsymbol{k}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}}{2\psi} + \frac{\boldsymbol{\omega}\times\hat{\mathbf{p}}}{2}$$

in a range of parameters:

$$\boldsymbol{v} \equiv \boldsymbol{u} + (\phi\psi\omega_y, -\phi\psi\omega_x, \phi)$$

gives the same physics: clustering in downwelling flow lowering rise velocity of bubbles, columnar structures

Solving for multifractals formed by non-tracer particles in turbulence

- Find limits where motion of small particles transported by turbulence is approximately smooth system $\dot{x} = v(t, x(t))$
- v differs from driving turbulent flow u: inertia, finite size, active swimming behavior, interactions with local scalar gradients (phoresis: temperature, salinity, attractant concentration, electric potential…) or vorticity (bubbles)…
- v is compressible: universal, independent of details of v, implications – multifractal distribution on small scales
- Flow definability demands often small compressibility
- Use theory of attractors of weakly dissipative systems (IF 2012) similar to near-equilibrium statistical physics including where the flow cannot be written explicitly.

Solving for multifractals formed by non-tracer particles in turbulence, cont.

- Small compressibility: lognormal density with correlation codimension twice the information codimension (IF 2012). Discrete distribution is Poisson with lognormal intensity (Schmidt, Holzner, IF 2017).
- Generalized fractal dimensions of attractors of a general smooth system can be found via large deviations function describing fluctuations of finite-time Lyapunov exponents
- Inhomogeneity of turbulence influences particles' distribution most profoundly: localization-delocalization transition, superexponential growth of density
- Multifractals in non-smooth flow: inertial range of supersonic turbulence, fluid density itself is multifractal

Supersonic turbulence: multifractals in the inertial range

Coincidence of active fluid density and tracers concentration:

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0, \quad \partial_t n + \nabla \cdot (n \boldsymbol{v}) = 0$$

So far believed passive concentration n agrees with active ρ:

$$\rho \left(\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla) \boldsymbol{v} \right) = -c_s^2 \nabla \rho + \rho \boldsymbol{a} + \nabla \sigma'$$

Can passive n differ from active ρ? (IF, Mond 2019 arxiv)

How could the difference come about? Steady state not unique. Mixing (decay of initial condition with zero integral):

$$\lim_{t \to \infty} \left(\frac{n(t, \boldsymbol{x})}{\int n(\boldsymbol{x}) d\boldsymbol{x}} - \frac{\rho(t, \boldsymbol{x})}{\int \rho d\boldsymbol{x}} \right) = 0$$

not evident because of back reaction! Density can form fine steady state which is stable due to interaction with flow however would not be stable for n (cf. vorticity/passive scalar)

Supersonic turbulence: multifractals in the inertial range, cont.

We propose a number of independent predictions for active fluid density and passive tracers concentration. The statistics of density is lognormal in isothermal case and complete spectrum of fractal dimensions (later) can be derived. For concentration we describe transition to multifractality as the Mach number increases. Scaling of potential and solenoidal components becomes similar so compression and stretching by the flow scales identically (the flow is self-similar disregarding intermittency). Stationarity condition on pair correlation function f(r) holds for the Navier-Stokes flow:

$$f(\boldsymbol{r}) \approx \int P(\boldsymbol{r}, \boldsymbol{r}', t) f(\boldsymbol{r}') d\boldsymbol{r}'$$

Pair correlation function of concentration

Stationarity condition for studying correlation dimension

$$f(r) = \left(\frac{\tilde{L}}{r}\right)^{\beta}, \quad r \ll L$$

We find the counterpart of result for smooth systems

$$\tilde{P}(r, r', t) \propto r^{-\beta}; \quad r, r' \ll L, \quad t \ll t_L$$

Transition to multifractality: Kraichnan model of propagator

$$\langle u_i(t_1, \boldsymbol{x}_1) u_k(t_2, \boldsymbol{x}_2) \rangle = \delta(t_2 - t_1) \left[V_0 \delta_{ik} - K_{ik}(\boldsymbol{r}) \right]$$

where eddy diffusivity tensor obeys

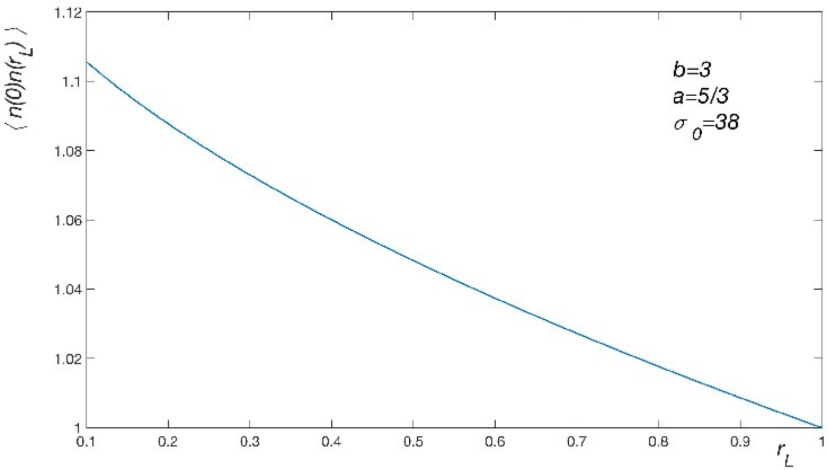
$$2K_{ik} = \left[\frac{(r^4 u)'}{r^3} - c\right] r^2 \delta_{ik} - \left[\frac{(r^2 u)'}{r} - c\right] r_i r_k$$

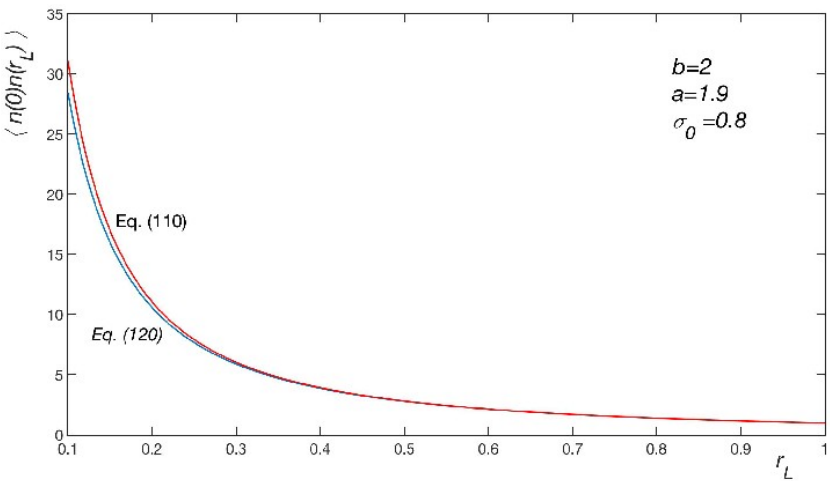
where c is potential and u solenoidal components

Pair correlation and zero modes

Kraichnan model describes transition to multifractality

$$\langle n(0)n(\boldsymbol{r})\rangle = \exp\left[\int_r^\infty \frac{c(r')dr'}{r'u(r')}\right]$$





Higher order correlations describe multifractality given by scalings of zero modes of $\sum_{nl}\nabla_{r_n^i}\nabla_{r_l^k}K_{ik}(\boldsymbol{r}_n-\boldsymbol{r}_l)$ as in anomalous scaling of passive scalar (Bernard, Gawedzki, Kupiainen and Falkovich, Kolokolov, Chertkov, Lebedev 1996). Multifractality accelerates astrophysical processes: formation of planetesimals and others

Input from fluid mechanics to dynamical systems

Distributions of particles in fluid mechanics demand and brought progress in smooth systems $\dot{x} = v(t, x(t))$:

- representation of finite time Lyapunov exponents, providing eigenvalues of Jacobi matrix as integrals over scalar noises: gives Oseledets, central limit theorems and large deviations theory
- formula for the sum of Lyapunov exponents
- large deviations for backward in time flow and GC relation
- lognormality of distribution and generalized fractal dimensions at small dissipation (compressibility)
- generalized fractal dimensions via the large deviations function and new results on dimension function

Multifractal formalism

- particles' density evolves by $\partial_t n + \nabla\cdot(n\boldsymbol{v}) = 0$ to a singular distribution $n_s(\boldsymbol{x})$ vanishing outside multifractal
- density divergence from $m_\epsilon(\boldsymbol{x}) = \int_{|\boldsymbol{x}'-\boldsymbol{x}|<\epsilon} n_s(\boldsymbol{x}')d\boldsymbol{x}'$

$$n_\epsilon(\boldsymbol{x}) \sim \epsilon^{\alpha(\boldsymbol{x})-d}, \quad \alpha(\boldsymbol{x}) = \lim_{\epsilon\to 0}\alpha_\epsilon(\boldsymbol{x}), \quad \alpha_\epsilon \equiv \frac{\ln m_\epsilon(\boldsymbol{x})}{\ln\epsilon}$$

- the attractor are interwoven fractal sets

$X = \hat{X} \cup \left(\bigcup_\alpha X_\alpha\right)$; X_α level set $\alpha(\boldsymbol{x}) = \alpha$, dim $f(\alpha)$

(Halsey, Jensen, Kadanoff, Procaccia, Shraiman 1986).

- randomly placed ball with radius Ɛ interesects X_α with probability $\epsilon^{d-f(\alpha)}$ giving (Frisch, Parisi 1985)

$$\int m_\epsilon^k(\boldsymbol{x})d\boldsymbol{x} = \int_{\alpha_1}^{\alpha_2} \epsilon^{k\alpha+d-f(\alpha)} h(\alpha)d\alpha \sim \epsilon^{\xi(k-1)+d}$$

$$\xi(k-1) = \min_\alpha [k\alpha - f(\alpha)]$$

Rényi and Hentschel-Procaccia dims

- Legendre transform of dimensions function $f(\alpha)$ defines continuous Rényi dimensions $\tilde{D}(k) \equiv \xi(k-1)/(k-1)$. These agree for k>0 with Hentschel-Procaccia dims:

$$D(k) \equiv \lim_{\epsilon \to 0} \frac{\ln \int m_\epsilon^{k-1}(\boldsymbol{x}) n_s(\boldsymbol{x}) d\boldsymbol{x}}{(k-1)\ln \epsilon}, \quad \langle m_\epsilon^k \rangle \sim \epsilon^{\xi(k)}$$

- it can be demonstrated that mass fluctuations obey

$$P_\epsilon(\alpha) = \int \delta\left(\alpha_\epsilon(\boldsymbol{x}) - \alpha\right) n_s(\boldsymbol{x}) d\boldsymbol{x} \sim \epsilon^{\alpha - f(\alpha)}$$

$$\alpha_\epsilon \equiv \frac{\ln m_\epsilon(\boldsymbol{x})}{\ln \epsilon}$$

- $f(\alpha)$ or its Legendre transform D(k) provide complete information on fluctuations of mass in space/on attractor

How to study attractors: Jacobi matrix and evolution of small volumes

- formation of density singularities is a small-scale phenomenon below smoothness (Kolmogorov) scale of v.
- it demands indefinite approach of trajectories describable by behavior of infinitesimally separated trajectories – Jacobi matrix $W_{ik}(t, \boldsymbol{x}) = \partial_k q_i(t, \boldsymbol{x})$ where $\boldsymbol{q}(t, \boldsymbol{x})$ are particles' trajectories labeled by position x at t=0
- local structure of attractors of dynamical systems must be invariant under W – key to the problem

Solvable limit: small compressibility

Small compressibility limit is singular: for pair correlations

$$f_1(r) \sim r^{D(2)-d}$$

at zero compressibility f=1, at non-zero f(0) diverges. However there's small δ for which $f_1(\delta) \approx 1$: the distribution is uniform at small finite scale. This gives

$$n_\epsilon(0,\boldsymbol{x}) = \exp\left(-\int_{-|\lambda_d|^{-1}\ln(\delta/\epsilon)}^{0} \nabla\cdot\boldsymbol{v}(t,\boldsymbol{q}(t,\boldsymbol{x}))dt\right)$$

which is explicit solution combining instantaneous flow with its temporal properties (IF 2012). Lognormality:

$$D(k) = d - C_{KY}k$$

(inconsistency large k Mandelbrot 1981) KY codimension

$$C_{KY} = |\sum_{i=1}^{d} \lambda_i/\lambda_d|$$

it agrees with small compressibility limit of 2d results of Gawedzki, Bec, Horvai 2004. Large k - optimal fluctuation

Sum of Lyapunov exponents and small compressibility limit

Usually Lyapunov exponents do not admit simple form via the flow. In contrast for the sum (with Falkovich 2004)

$$\sum_{i=1}^{d}\lambda_i=-\int_0^{\infty}\langle\nabla\cdot\boldsymbol{v}(0)\nabla\cdot\boldsymbol{v}(t)\rangle dt$$

can be considered as FDT arbitrarily far from equilibrium since sum is entropy production rate in non-equilbrium ss.

$$\langle n(0)n(\boldsymbol{r})\rangle=\langle n\rangle^2\left(\frac{\eta}{r}\right)^{\Delta}$$

Pair correlation entering collision kernel obeys

$$\Delta=\frac{1}{|\lambda_d|}\int_{-\infty}^{\infty}\langle\nabla\cdot\boldsymbol{v}(0)\nabla\cdot\boldsymbol{v}(t)\rangle dt$$

where Lyapunov exponent and trajectories in the correlation function are of incompressible flow (~equilib)

Optimal fluctuation

Compressibility is never small for large k (with Balkovsky, Falkovich 2001; Gawedzki, Bec, Horvai).

$$\langle m_l^k \rangle = \int_0^1 m_l^k P(m_l) dm_l$$

There is optimal fluctuation that provides maximal order one mass in Ɛ ball by compression from all sides (with Zhou 2018):

$$\eta \exp\left(\sigma_1(t^*) t^*\right) \sim \epsilon$$

this gives

$$\langle m_\epsilon^k \rangle \sim \int_{\sigma_1<0, \quad \sigma_i \geq \sigma_{i+1}} \exp\left(-t^* H(\sigma_1, .., \sigma_d)\right) d\boldsymbol{\sigma}$$

producing

$$\xi_k = kD(k+1) = -\max_{\sigma_1<0, \quad \sigma_i \geq \sigma_{i+1}} \frac{H(\boldsymbol{\sigma})}{\sigma_1}, \quad k > k_*$$

Threshold need not be large – it is of order one for compressibility of order one. Can include D(2)!

Stationarity condition for D(k) and partial dimensions

Mass conservation $m_\epsilon(t, \boldsymbol{q}(t, \boldsymbol{x})) = m_{\epsilon e^{-t\sigma_1}, \ldots, \epsilon e^{-t\sigma_d}}(\boldsymbol{x}, 0)$
produces (Grassberger, Badii, Politi; Gawedzki, Bec, Horvai)

$$\int m_\epsilon^k(\boldsymbol{y}, t) n(\boldsymbol{y}, t) d\boldsymbol{y} = \int m^k_{\epsilon e^{-t\sigma_1}, \ldots, \epsilon e^{-t\sigma_d}}(\boldsymbol{x}, 0) n(\boldsymbol{x}, 0) d\boldsymbol{x}$$

which gives stationarity condition on partial dimensions:

$$\langle \exp\left(-tk\boldsymbol{\sigma} \cdot \boldsymbol{D}(k+1)\right)\rangle \sim 1$$

where D(k) is sum of D_i(k). How to get vector from one condition?

It can be seen that D(k) is maximal for **D**(k)=(1,..,1, $\delta(k)$, 0,..,0) where $\delta(k)$ is fractional part of the dimension and 1

appears n(k) times where n(k) is integer part of D(k).

KY-type "maximal disorder" assumption for D(k)

We assume with Zhou (2018, unpublished) that D(k) is given by its maximal possible value compatible with stationarity condition. This gives

$$\min_{\sigma_i \geq \sigma_{i+1}} \left[k \sum_{i=1}^{n(k+1)} \sigma_i + k\delta(k+1)\sigma_{n(k+1)+1} + H(\boldsymbol{\sigma}) \right] = 0$$

This formula is a KY-type assumption that locally the attractor is confined to a hypersurface which is a product of continua and a Cantor set with fractional dimension (Ledrappier Young 1988 proved KY conjecture in random case). Independently arrived at by Gawedzki.

Consistency and implications

The conjecture agrees with all that is known: for information dimension D(1) it reproduces Kaplan-Yorke conjecture, for D(2) it reproduces Baxendale formula, it reproduces small compressibility limit and optimal fluctuation result at large k. For flows that preserve total volume it gives box-counting dimension D(0)=d, seen in some previous cases. It refines Grassberger-Procaccia:

$$\alpha = -\frac{\boldsymbol{D}_{KY}\Gamma\boldsymbol{D}_{KY}}{2|\lambda_{k+1}|}$$

where α=D’(1) and Γ is covariance matrix in CLT for Lyapunovs. Using back in time Lyapunov exponents:

$$D'(0) = \sum_{i=1}^{d} \tilde{\lambda}_i/\tilde{\lambda}_1$$

Inhomogeneous turbulence (channel)

The above description holds far from the boundaries (neglecting stratification). In inhomogeneous turbulence clustering is different (unless really small scales are considered). For time-dependent problem there is self-acceleration where the volume continuously enters regions with stronger compressibility which results in superexponenential growth predicted theoretically and confirmed by DNS in channel turbulence (with Schmidt, Ditlevsen, van Reeuwijk, Holzner 2018). Description of the steady state is the current study.

Open problems

One-way coupling: discrete distribution of particles' number in small volume on attractor, inhomogeneous turbulence
Two-way coupling: interaction of flow with particles distributed on a multifractal
Supersonic turbulence: is there counterpart of Kaplan-Yorke formula for information dimension? can generalized fractal dimensions of supersonic turbulence be predicted?
Lognormality of isothermal case derived? Implications of mulifractality of density of the fluid itself in the limit of infinite Mach number?