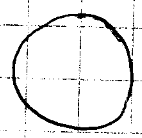
Knots and Braids

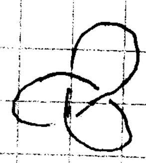
references: Jones: ~~Hecke algebra representations of braid groups and link polynomials~~

~~books~~ — Ohtsuki: Quantum Invariants
Kauffman: Knots and Physics
Kassel, Turaev: Braid Groups

def: A knot is an embedding of S^1 into $\mathbb{R}^3$ or into S^3. A link is an embedding of $S^1 \cup \ldots \cup S^1$.



trivial knot



trefoil knot

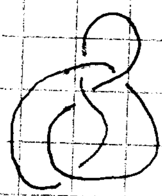
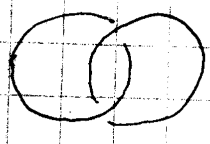


figure-eight knot



Hopf link

Two knots (or two links) K and K' are called isotopic if there exists a smooth (or piecewise smooth) family of homeomorphisms $h_t : \mathbb{R}^3 \to \mathbb{R}^3$ for $t \in [0,1]$ such that h_0 is the identity map of $\mathbb{R}^3$ and $h_1(K) = K'$.

In other words, K and K' are isotopic if K is obtained from K' by a continuous deformation such that there is no self-intersection at any time during the deformation.

easy to show two knots are isotopic → find an explicit step-by-step process of the deformation between the two knots

to show that two knots are not isotopic one needs knot invariants

For a well-known set S we call the map

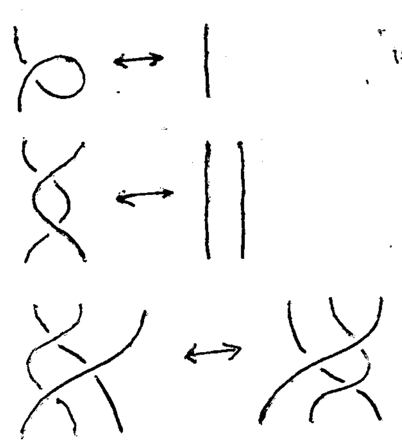
$$I : \{\text{knots}\} \longrightarrow S$$

an isotopy invariant of knots, if the map satisfies $I(K) = I(K')$ for any two isotopic knots K and K'.

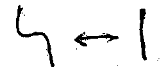
thm : (Reidemeister) Let K and K' be two knots (or two links, in general) and D and D' diagrams of them. Then, K is isotopic to K' in $\mathbb{R}^3$ if and only if D is related to D' by a sequence of isotopies of $\mathbb{R}^2$ and the RI, RII, RIII moves below.

Reidemeister moves

RI



isotopy



RII

RIII

proof : It is trivial to show that, if D and D' are related by a sequence of the moves, then K and K' are isotopic

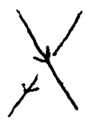
$\leftarrow$ showing K, K' isotopic $\rightarrow$ they are related by the Reidemeister moves is more complicated

$$\{\text{knots}\} / \text{isotopy of } \mathbb{R}^3 = \{\text{knot diagrams}\} / \text{RI, RII, RIII and isotopy of } \mathbb{R}^2$$

ex ~~[illegible]~~ linking number

use oriented links

positive crossing

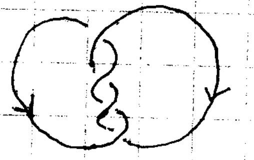


negative crossing

Knots and Braids

The linking number of two components L_1 and L_2 of an oriented link is defined by

$$lk(L_1, L_2) = \frac{1}{2}\Big(\big(\text{the number of positive crossings of two strands of } D_1 \text{ and } D_2\big) - \big(\text{the number of negative crossings of two strands of } D_1 \text{ and } D_2\big)\Big)$$



linking number 2

Prop : The linking number $lk(L_1, L_2)$ is an isotopy invariant of an oriented link $L_1 \cup L_2$.

aside : The linking number was introduced by Gauss in the form of the linking integral while studying electromagnetism.

For differentiable curves $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ define

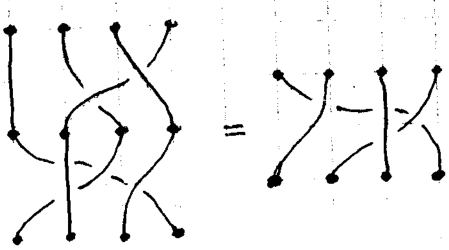
$$\text{linking number} = \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\bar{r}_1 - \bar{r}_2}{|\bar{r}_1 - \bar{r}_2|^3} \cdot (d\bar{r}_1 \times d\bar{r}_2)$$

A braid in n strands is a union of n strands embedded in $\mathbb{R}^2 \times [0,1]$ such that its boundary is the set $\{1, 2, \ldots, n\} \times \{0\} \times \{0,1\}$ in $\mathbb{R}^2 \times [0,1]$ and such that no strand has a critical point with respect to the vertical coordinate

a braid

Two braids are isotopic if they are related by an isotopy of $\mathbb{R}^2 \times [0,1]$ preserving its boundary and the vertical coordinate

The isotopy classes of braids (in n strands) form a group B_n where the product is given by concatenation



$b_1 b_2$

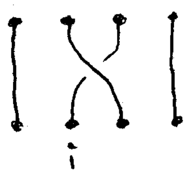
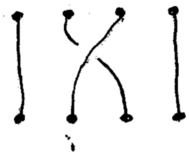
<u>def</u>: The Artin braid group B_n is the group generated by $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the braid relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

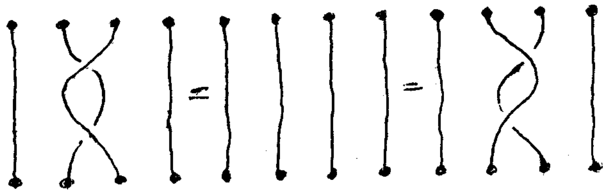
for all $i, j = 1, 2, \ldots, n-1$ with $|i-j| \geq 2$, and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

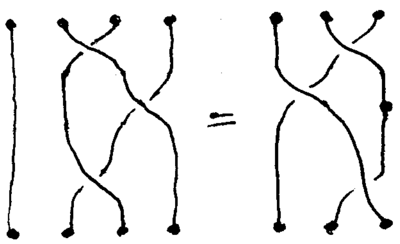
for $i = 1, 2, \ldots, n-2$.

σ_i  σ_i^{-1} 

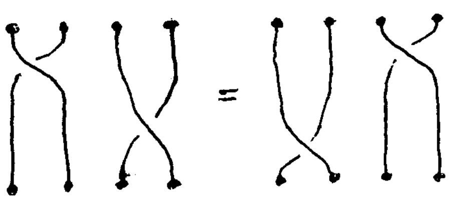
$\sigma_i^{-1}\sigma_i = e = \sigma_i \sigma_i^{-1}$



$\sigma_i \sigma_{i+1} \sigma_i$
$= \sigma_{i+1} \sigma_i \sigma_{i+1}$



$\sigma_i \sigma_j = \sigma_j \sigma_i$
$|i-j| \geq 2$



It is easy to show that the relations of the braid group are derived from isotopy and Reidemeister moves. Conversely, it is not so easy to show that the above relations are sufficient; it can be shown ~~by applying~~ similarly to the Reidemeister thm

Knots and Braids

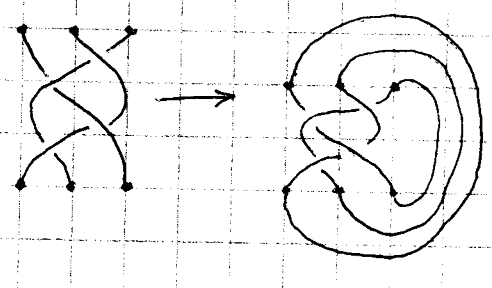
comparison with the symmetric group:

The symmetric group S_n with $n \geq 1$ is the group of all permutations of the set $\{1, 2, \dots, n\}$.
The group law of S_n is the composition of permutations, and the neutral element is the identity permutation that fixes all elements of $\{1, 2, \dots, n\}$.
Denote by $\tau_{i,j}$ the permutation exchanging i and j and leaving the other elements of $\{1, 2, \dots, n\}$ fixed. Such a permutation is called a transposition.
When $j = i+1$, we write s_i for $\tau_{i,j}$. The transpositions $s_1, \dots, s_{n-1}$ are called simple transpositions. The simple transpositions satisfy the following relations for all $i, j = 1, \dots, n-1$

$$s_i s_j = s_j s_i \quad \text{if } |i-j| \geq 2$$
$$s_i s_j s_i = s_j s_i s_j \quad \text{if } |i-j| = 1$$
$$s_i^2 = 1$$

There is a surjective group homomorphism $B_n \to S_n$ such that $b_i \mapsto s_i$. In other words the symmetric group is isomorphic to a quotient of the braid group.

The closure of a braid is the link obtained from the braid by connecting upper ends and lower ends respectively

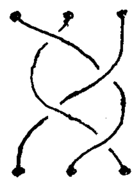


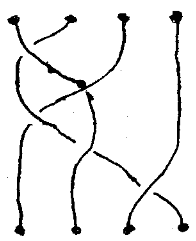
<u>thm</u> (Alexander): Any (oriented) link is isotopic to the closure of some braid (with downward orientation).

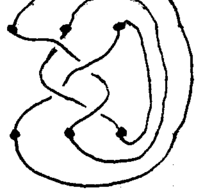
consider the map
$$\bigcup_{n \geq 1}^{\infty} B_n \longrightarrow \{\text{oriented links}\} / \text{isotopic}$$
$$\text{braid} \mapsto \text{closure of the braid}$$

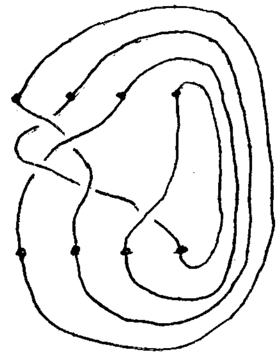
we have just seen that this map is surjective, but it is not 1-1

There are some simple ways to modify braids so that their closures are isotopic links. First there is the Markov move: suppose β is a braid word in B_n. Then the three braids β, $\beta\sigma_n$ and $\beta\sigma_n^{-1}$ all have isotopic closures. For example,

$\beta =$  $= \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1} \in B_3$

$\beta\sigma_3^{-1}$  $\in B_4$

$\overline{\beta} =$ 

$\overline{\beta\sigma_3^{-1}} =$ 

Also the closure of $g\beta g^{-1}$ for any g in B_n is isotopic to the closure of β

<u>thm</u>: (Markov) Let b and b' be two braids, and L and L' their closures. Then, L is isotopic to L' as oriented links if and only if b is related to b' by a sequence of the following MI and MII moves,

MI: $ab \leftrightarrow ba$ for any $a, b \in B_n$

MII: $b\sigma_n \leftrightarrow b \leftrightarrow b\sigma_n^{-1}$ for any $b \in B_n$

where we regard the b of $b\sigma_n^{\pm 1}$ in the MII move as the braid in B_{n+1} obtained from the original by by adding a trivial $(n+1)$-st strand.

MI, MII are called Markov moves

Knots and Braids

$$\{\text{oriented links}\}/\text{isotopy} = \left(\bigcup_{n=1}^{\infty} B_n\right)/\text{the MI, MII moves}$$

isotopy invariants of oriented links in $\mathbb{R}^3$ $\longleftrightarrow$ functions on $\coprod_n B_n$ constant on the Markov equivalence classes

<u>def</u>: A Markov function with values in a set E is a sequence of set-theoretic maps $\{f_n : B_n \to E\}_{n\geq 1}$, satisfying the following conditions:

i) for all $n \geq 1$ and all $\alpha, \beta \in B_n$

$$f_n(\alpha\beta) = f_n(\beta\alpha)$$

ii) for all $n \geq 1$ and all $\beta \in B_n$

$$f_n(\beta) = f_{n+1}(\beta\sigma_n) \quad \text{and} \quad f_n(\beta) = f_{n+1}(\beta\sigma_n^{-1})$$

Condition i) is automatically satisfied by the trace function so an obvious place to look for Markov functions is to find linear representations of the braid groups in such a way that the characters are compatible with ii)

<u>examples</u> the reduced Burau representation

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 & \cdots & 0 \\ 1 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \qquad \sigma_i \mapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & 1 & t & 0 & \\ & 0 & -t & 0 & \\ & 0 & 1 & 1 & \\ & & & & \ddots \; 1 \end{pmatrix} \qquad \sigma_{n-1} \mapsto \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & t \\ & & 0 & -t \end{pmatrix}$$

<u>thm</u>: Let $\beta(\alpha)$ be the Burau matrix of $\alpha \in B_n$. Then

$$\det(1 - \beta(\alpha)) = (1 + t + \cdots + t^{n-1})\, \Delta_{\tilde{\alpha}}(t)$$ where $\Delta_{\tilde{\alpha}}(t)$ is the Alexander polynomial of $\tilde{\alpha}$.

2) Hecke algebra $H(q,n)$ with presentation on generators $g_1, g_2, \ldots, g_{n-1}$ and relations

$$g_i^2 = (q-1)g_i + q$$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

$$g_i g_j = g_j g_i \qquad \text{if } |i-j| \geq 2$$

Ocneanu $\to$ there is a Markov trace on $H(z,n)$

def: The two-variable invariant $X_L(z,\lambda)$ of the oriented link L is the function

(HOMFLY) $$X_L(z,\lambda) = \left(-\frac{1-\lambda z}{\sqrt{\lambda}(1-z)}\right)^{n-1} (\sqrt{\lambda})^e \, tr(\pi(\alpha))$$

where $\alpha \in B_n$ is any braid with $\hat{\alpha} = L$ and π is the representation of B_n in $H(z,n)$, $\sigma_i \mapsto g_i$.

3) Temperley-Lieb algebra A_n generators $1, e_1, e_2, \dots, e_n$ and relations

$$e_i^2 = e_i \qquad e_i e_{i\pm1} e_i = \tau e_i \qquad e_i e_j = e_j e_i \quad \text{if } |i-j| \geq 2$$

A_n is a quotient of $H(t, n+1)$ and has a Markov trace

$\alpha \in B_n$

Jones polynomial $-$ $$V_L(t) = \left(-\frac{t+1}{\sqrt{t}}\right)^{n-1} (\sqrt{t})^e \, tr(\pi_0(\alpha))$$

where π_0 is a representation of B_n into A_{n-1} given by $\pi_0(\sigma_i) = t e_i - (1-e_i)$