A CONCRETE CASE OF A q-DEFORMED SCHUR-WEYL DUALITY

I. THE TEMPERLEY-LIEB ALGEBRA

While studying knot invariants we encountered the following algebras.

<u>Def</u>: For $\nu \in \mathbb{C}$ and $N \in \mathbb{N}$, $TL_N^{(\nu)}$ is the algebra with generators

$$e_1, e_2, e_3, \ldots\ldots, e_{N-2}, e_{N-1}$$

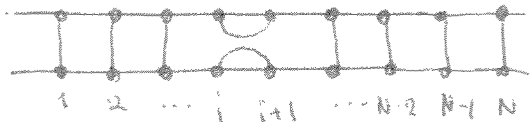
and relations

$$e_j^2 = \nu\, e_j \qquad (\text{for all } j)$$

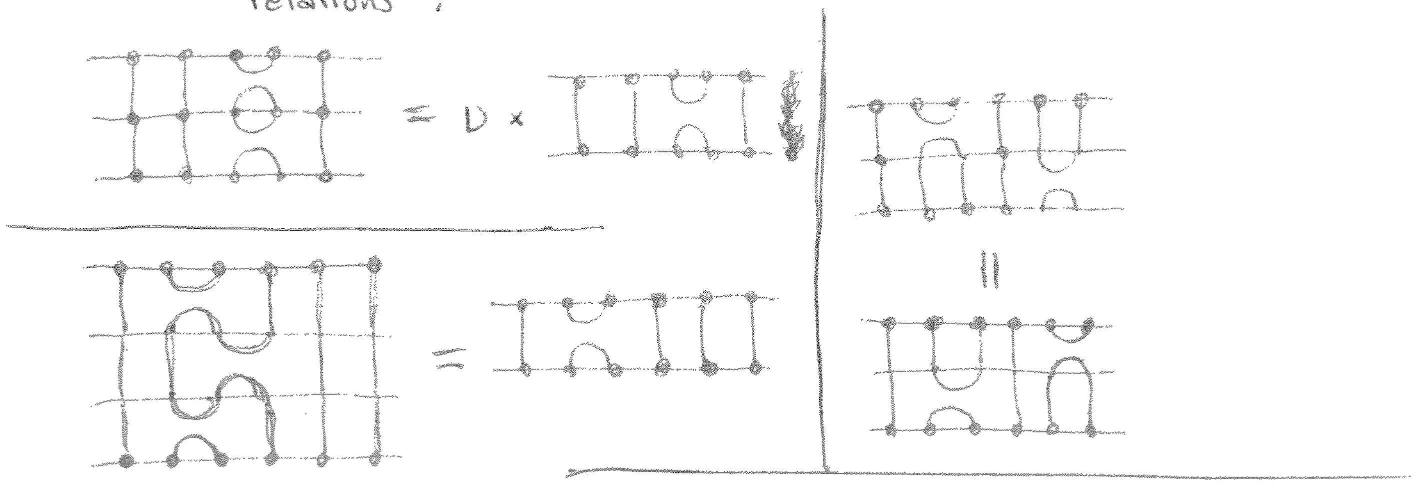
$$e_j e_{j\pm1} e_j = e_j \qquad (\text{for } j, j\pm1 \in \{1,2,\ldots,N-1\})$$

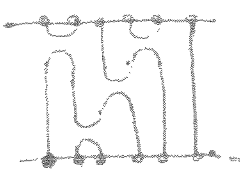
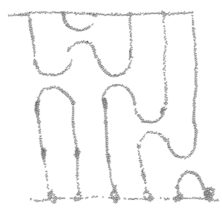
$$e_j e_k = e_k e_j \qquad (\text{for } |j-k|>1).$$

We visualize this as an algebra of diagrams (up to isotopy) with concatenation product

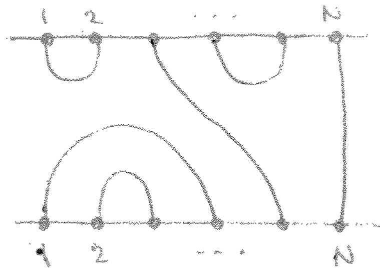
generators : $e_j =$ 

relations :



typical elements : $a\,\cdot$  $+\, b$ 

It can be shown that a basis of $TL_N^{(\nu)}$ is obtained from all "diagrams"

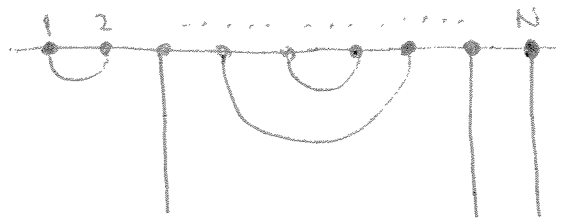


where N points at the bottom and N points at the top are pairwise joined by N curves which don't cross each other.

The dimension is therefore a Catalan number (see below)

Link state representations

Let $\mathcal{M}_p^N$ be the vector space with basis indexed by "(N,p) - link states"

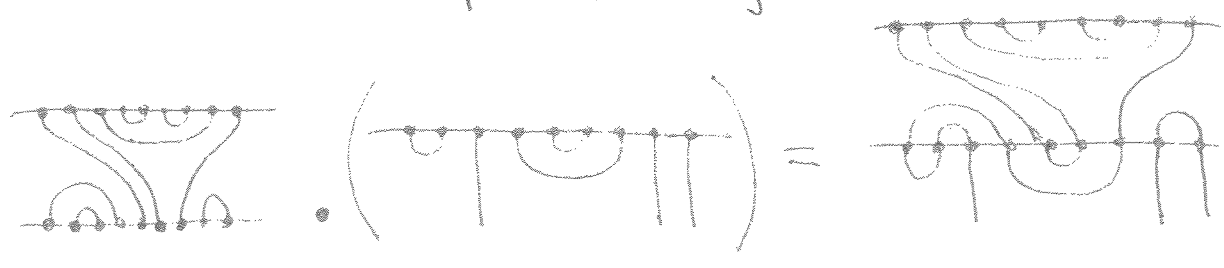


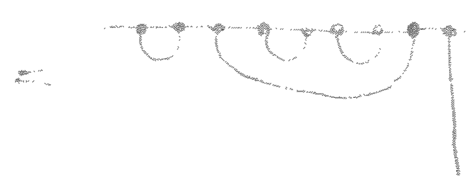
$(N=9\ ,\ p=3)$

where N points at the top have p curves connecting pairs of points and $N-2p$ points joined to infinity all with noncrossing curves in the lower half-plane.

$$\mathcal{M}^N = \mathcal{M}_0^N \oplus \mathcal{M}_1^N \oplus \mathcal{M}_2^N \oplus \cdots \oplus \mathcal{M}_{\lfloor N/2 \rfloor}^N \qquad \text{(as vector space)}$$

The space $\mathcal{M}^N$ becomes a representation of $TL_N^{(\nu)}$ by "concatenation from the top" (and factor ν for all disconnected loops), e.g.





"the full link state representation"

Note: the action of $TL_N^{(\nu)}$ never reduces the number p of "paired" points, so we have nested submodules

$$\mathcal{M}^N = \bigoplus_{p=0}^{\lfloor N/2 \rfloor} \mathcal{M}_p^N \supset \bigoplus_{p=1}^{\lfloor N/2 \rfloor} \mathcal{M}_p^N \supset \bigoplus_{p=2}^{\lfloor N/2 \rfloor} \mathcal{M}_p^N \supset \cdots \supset \mathcal{M}^N_{\lfloor N/2 \rfloor}$$

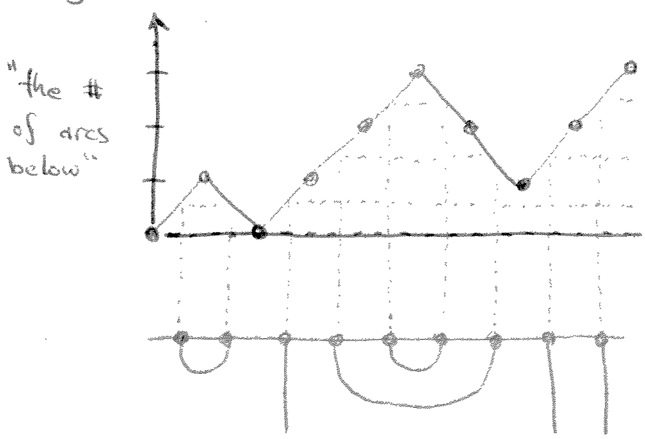
We call the consecutive quotients

$$\mathcal{V}_p^N = \Big(\bigoplus_{p'=p}^{\lfloor N/2 \rfloor} \mathcal{M}_{p'}^N \Big) \Big/ \Big(\bigoplus_{p'=p+1}^{\lfloor N/2 \rfloor} \mathcal{M}_{p'}^N \Big)$$

the (N,p)-link state representations ($\underline{\text{NOTE}}$: $\dim \mathcal{V}_p^N = \dim \mathcal{M}_p^N$)

$\underline{\text{Fact}}$ For ν not of the form $2\cdot\cos(\pi \frac{m}{n})$ the representations $\mathcal{V}_p^N$, $p = 0,1,2,\ldots,\lfloor N/2 \rfloor$, are distinct (i.e. non-isom.) irreducible representations of $TL_N^{(\nu)}$.

A bijection between link states and walks:

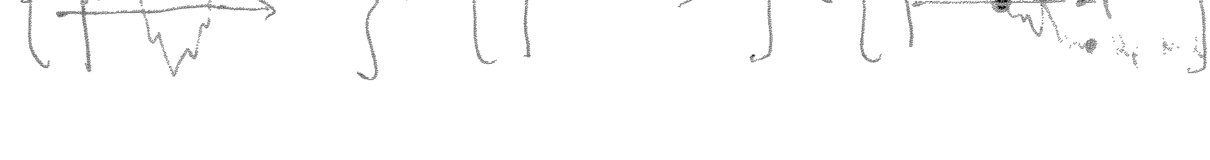


So:

$$\dim(\mathcal{V}_p^N) = \#\{\omega \in \mathbb{N}^{\{0,1,2,\ldots,N\}} : \omega_0 = 0, \ \omega_N = N-2p, \ |\omega_j - \omega_{j-1}| = 1\}$$

Counting the walks by reflection principle

- clearly $\#\{\omega \in \mathbb{Z}^{\{0,1,\ldots,N\}} : \omega_0 = 0, \ \omega_N = N-2p, \ |\omega_j-\omega_{j-1}|=1\}$ is $\binom{N}{p}$, the choice of positions of down-steps
- reflect across level -1 if a walk is not non-negative

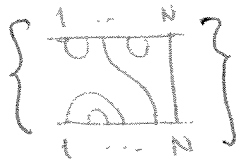
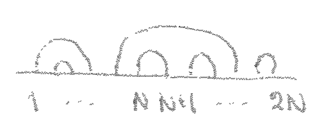


$\{\text{all } N\text{-step walks to } N-2p\} = \{\text{non-neg. walks to } N-2p\} \sqcup \{\text{walks to } N-2p \text{ which visit } -1\}$

$$\Rightarrow \binom{N}{p} = \dim(\mathcal{V}_p^N) + \binom{N}{p-1}$$

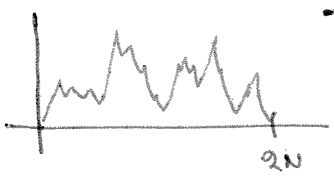
$$\Rightarrow w_p^N = \dim(\mathcal{V}_p^N) = \binom{N}{p} - \binom{N}{p-1} = \frac{1}{p+1}\binom{N}{p}$$

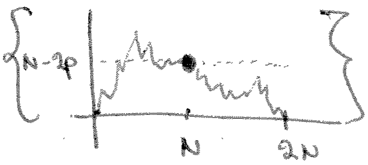
Note also the dimension of $TL_N^{(\nu)}$

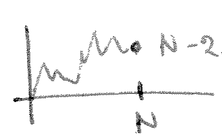
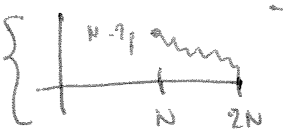
$\dim TL_N^{(\nu)} = \# \{$  $\} = \# \{$  $\}$

$= \# \{\text{non-neg. walks of } 2N \text{ steps to } 0\} = w_N^{2N}$

Sum of squares formula:

$\dim TL_N^{(\nu)} = \# \{$  $\}$

$= \# \bigsqcup_{p=0}^{\lfloor N/2 \rfloor} \{$  $\}$

$= \# \bigsqcup_{p=0}^{\lfloor N/2 \rfloor} \left(\{\right.$  $\} \times \{$  $\left.\}\right)$

$$= \sum_{p=0}^{\lfloor N/2 \rfloor} (w_p^N)^2 = \sum_{p=0}^{\lfloor N/2 \rfloor} \left(\dim \mathcal{V}_p^N\right)^2$$

Corollary: For ν not of the form $2\cos(\pi \frac{m}{n})$ the algebra $TL_N^{(\nu)}$ is semisimple (all finite-dim. modules are direct sums of irreps)

Proof: $\mathcal{V}_p^N$ are distinct irreps, sum of squares of dimensions is the dimension of the algebra. Conclude by Wedderburn's structure theorem.

II. TENSOR POWERS OF THE STANDARD REPRESENTATION OF $\mathcal{U}_q(\underline{sl}_2)$

Let $A = \mathcal{U}_q(\underline{sl}_2)$ be the algebra with generators

$$E, F, K, K^{-1}$$

and relations

(Here ~~[illegible]~~ $q \in \mathbb{C} \setminus \{0, +1, -1\}$)

$$KEK^{-1} = q^2 E$$

$$KFK^{-1} = q^{-2} E$$

$$KK^{-1} = 1 = K^{-1}K$$

$$EF - FE = \frac{1}{q - q^{-1}} (K - K^{-1}).$$

(However, we often have to assume that $q^n \neq 1$ for all $n = 1, 2, 3, \ldots$)

Define

$$[m]_q = q^{m-1} + q^{m-3} + q^{m-5} + \cdots + q^{3-m} + q^{1-m} = \frac{q^m - q^{-m}}{q - q^{-1}}$$

for $m \in \mathbb{Z}$ (NOTE: $\lim_{q \to 1} [m]_q = m$)

and $[m]_q! = [m]_q \cdot [m-1]_q \cdots [2]_q \cdot [1]_q$.

Remark: If p is such that $q^p = 1$, then $[p]_q = 0$.

~~q is not a root of unity~~

For $d \in \mathbb{N}$ and $\varepsilon \in \{+1, -1\}$ we have a representation $W_d^{(\varepsilon)}$ of A with basis $w_0, w_1, \ldots, w_{d-1}$ and action of the generators

$$K . w_j = \varepsilon q^{d-1-2j} w_j$$

$$F . w_j = w_{j+1} \qquad (F . w_{d-1} = 0)$$

$$E . w_j = \varepsilon \cdot [j]_q \cdot [d-j]_q \, w_{j-1} \qquad (E . w_0 = 0)$$

For q not a root of unity, $W_d^{(\varepsilon)}$ are all the distinct irreducible representations of A. Furthermore, A is semisimple for q not a root of 1. (any finite-dim A-module is a direct sum of irreps).

We give A the unique structure of a Hopf algebra such that the coproducts of the generators are

$$\Delta(K) = K \otimes K \qquad \Delta(F) = K^{-1} \otimes F + F \otimes \mathbb{1}$$
$$\Delta(E) = E \otimes K + \mathbb{1} \otimes E ,$$

which means that if U and V are A-modules, then $U \otimes V$ becomes an A-module via

$$K.(u \otimes v) = (Ku) \otimes (Kv)$$
$$E.(u \otimes v) = (Eu) \otimes (Kv) + u \otimes (Ev)$$
$$F.(u \otimes v) = (K^{-1}u) \otimes (Fv) + (Fu) \otimes v \quad .$$

We have for example $W_d^{(\varepsilon)} \cong W_d^{(+1)} \otimes W_1^{(\varepsilon)}$ and from here on we focus on $W_d = W_d^{(+1)}$ only.

The irreducible (for q not a root of 1) A-modules W_d are q-deformations of the irreducible d-dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$. We have for example

$$W_{d_1} \otimes W_{d_2} \cong W_{d_1+d_2-1} \oplus W_{d_1+d_2-3} \oplus \dots \oplus W_{|d_1-d_2|+3} \oplus W_{|d_1-d_2|+1}$$

(See e.g. Antti's model solutions, Probl. Sheet 12 : Exercise 3)
(Check dimensions : $d_1 d_2 = \sum_{k=|d_1-d_2|+1}^{d_1+d_2-1} k$ ok "by Gauss")

We call the q-deformation of the defining representation (2-dim.) of $\mathfrak{sl}_2$ the standard representation.

Inductively, the tensor powers of the standard rep. are

$$W_2^{\otimes N} = \underbrace{W_2 \otimes W_2 \otimes \dots \otimes W_2}_{N \text{ times}}$$
$$\cong w_0^N W_{N+1} \oplus w_1^N W_{N-1} \oplus w_2^N W_{N-3} \oplus \dots \oplus w_{\lfloor N/2 \rfloor}^N W_{\dots}$$

since $W_2 \otimes W_d \cong W_{d+1} \oplus W_{d-1}$.

Let's do the details in the case $W_2 \otimes W_2$

- basis of W_2 : w_0 and w_1 such that

$$K.w_0 = q\, w_0 \qquad E.w_0 = 0 \qquad F.w_0 = w_1$$
$$K.w_1 = q^{-1} w_1 \qquad E.w_1 = w_0 \qquad F.w_1 = 0$$

- the "triplet" : $t_0 = w_0 \otimes w_0 \in W_2^{\otimes 2}$ satisfies

$$E.t_0 = \underbrace{(Ew_0)}_{=0} \otimes (Kw_0) + w_0 \otimes \underbrace{(Ew_0)}_{=0} = 0$$
$$K.t_0 = \underbrace{(Kw_0)}_{=qw_0} \otimes \underbrace{(Kw_0)}_{=qw_0} = q^2\, w_0 \otimes w_0 = q^2\, t_0$$

Therefore $W_2 \otimes W_2$ contains a 3-dimensional irred. submodule spanned by

$$t_0 = w_0 \otimes w_0$$
$$t_1 = F.t_0 = (K^{-1}w_0) \otimes (Fw_0) + (Fw_0) \otimes w_0 = q^{-1} w_0 \otimes w_1 + w_1 \otimes w_0$$
$$t_2 = F^2.t_0 = q^{-1}(Fw_0) \otimes w_1 + (K^{-1}w_1) \otimes (Fw_0)$$
$$= (q^{-1} + q)\, w_1 \otimes w_1 = [2]_q\, w_1 \otimes w_1$$

- the "singlet" : $s = q\, w_0 \otimes w_1 - w_1 \otimes w_0 \in W_2^{\otimes 2}$ satisfies

$$E.s = q\, w_0 \otimes \underbrace{(Ew_1)}_{=w_0} - \underbrace{(Ew_1)}_{=w_0} \otimes \underbrace{(Kw_0)}_{=q w_0} = (q-q)\, w_0 \otimes w_0 = 0$$

$$K.s = s$$

Therefore s spans a one-dimensional (trivial, $\cong W_1$) submodule of $W_2 \otimes W_2$

- Recall : a submodule is a direct summand
  $\iff$ there exists a projection to the submodule which is an A-module homomorphism

Here, for example $W_2 \otimes W_2 \cong \underbrace{W_3}_{\mathrm{span}\{t_0, t_1, t_2\}} \oplus \underbrace{W_1}_{\mathrm{span}\{s\}}$

and the A-linear projection to $W_1 \subset W_2^{\otimes 2}$ is

$$\pi(w_0 \otimes w_0) = 0 \qquad \pi(w_1 \otimes w_1) = 0$$
$$\pi(w_0 \otimes w_1) = \frac{1}{q+q^{-1}}\, s \qquad \pi(w_1 \otimes w_0) = -\frac{q^{-1}}{q+q^{-1}}\, s$$

(Denote $\nu = -q - q^{-1}$)

A relation for consequtive projections:

in $W_2^{\otimes N}$ denote by π_j the projection to the "singlet" in j^{th} and $(j+1)^{st}$ components

$$W_2 \otimes W_2 \otimes \cdots \otimes \underbrace{(W_2 \otimes W_2)}_{\cong W_3 \oplus W_1} \otimes \cdots \otimes W_2 \longrightarrow W_2 \otimes \cdots \otimes \text{"span}\{s\}\text{"} \otimes \cdots \otimes W_2$$

<u>Lemma</u>:

$$\pi_j \circ \pi_k = \pi_k \circ \pi_j \quad \text{if } |j-k|>1 \quad \text{(OBVIOUS)}$$

$$\pi_j \circ \pi_j = \pi_j \quad \text{for all } j \quad \text{(PROJECTION)}$$

$$\pi_j \circ \pi_{j\pm 1} \circ \pi_j = \nu^2 \times \pi_j \quad (\nu = -q-q^{-1})$$

Proof: It suffices to do the calculation in $W_2 \otimes W_2 \otimes W_2$ for the last property. Schur's lemma and irreducibility of W_2 help. $\square$

A consequence: Letting $\nu = -q-q^{-1}$ and letting the generators e_j of $TL_N^{(\nu)}$ act by $\nu \times \pi_j$ defines a representation of $TL_N^{(\nu)}$ on $W_2^{\otimes N}$.

<u>Remark</u>: When q is not a root of unity, $\nu = -q-q^{-1}$ is not of the form $2\cos(\pi \frac{m}{n})$ so both A and $TL_N^{(\nu)}$ are semisimple.

III. THE QUANTUM SCHUR-WEYL DUALITY

The vector space $V = W_2^{\otimes N}$ carries both

- a representation of $A = \mathcal{U}_q(\mathfrak{sl}_2)$
  (via multiple coproducts and A-module structure of W_2)
- a representation of $TL_N^{(\nu)}$
  (via projections to singlets in consequtive factors)

and these actions commute with each other (any Temperley-Lieb element acts as an A-module map).

Assume q is not a root of unity, so both A and $TL_N^{(\nu)}$ (with $\nu = -q - q^{-1}$) are semisimple.

We get direct sum decompositions

(i) as A-modules

$$V \cong \bigoplus_{p=0}^{\lfloor N/2 \rfloor} \omega_p^N \, W_{1+N-2p}$$

multiplicity found inductively; $1+N-2p$ dimensional irreducibles

(ii) as $TL_N^{(\nu)}$-modules

$$V \cong \bigoplus_{p=0}^{\lfloor N/2 \rfloor} m_p^N \, \mathcal{V}_p^N$$

multiplicities? the (N,p)-link state rep. (irreducible) has dimension ω_p^N

Apply Schur's lemma

(i) to the A-module V and the A-module map given by the action of an element $x \in TL_N^{(\nu)}$:

- $x|_{W_{1+N-2p}}$ is zero or isomorphism

$\Rightarrow x.W_{1+N-2p} \subset \omega_p^N \, W_{1+N-2p}$ (these have the correct isomorphism class)

$\Rightarrow$ the submodules $\omega_p^N \, W_{1+N-2p} \subset V$ are stable under the action of x

$\therefore \omega_p^N \, W_{1+N-2p}$ is also a $TL_N^{(\nu)}$-submodule

(ii) to the $TL_N^{(\upsilon)}$ module $\mathcal{V}$ and the $TL_N^{(\upsilon)}$-module map given by an element $X \in A = \mathcal{U}_q(s\ell_2)$

- $X|_{\mathcal{V}_p^N}$ is zero or isomorphism

$\Rightarrow$ $X . \mathcal{V}_p^N \subset m_p^N \mathcal{V}_p^N$

$\Rightarrow$ the submodules $m_p^N \mathcal{V}_p^N \subset \mathcal{V}$ are stable under the action of X

$\therefore$ $m_p^N \mathcal{V}_p^N \subset \mathcal{V}$ is also an A-submodule.

One may wonder whether the "double submodules" (for A and $TL_N^{(\upsilon)}$) $w_p^N W_{1+N-2p}$ and $m_p^N \mathcal{V}_p^N$ coincide, especially given the observation that

$$\text{multiplicity of } W_{1+N-2p} = w_p^N = \dim(\mathcal{V}_p^N)$$

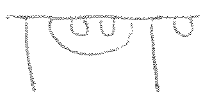
Indeed, they do coincide; and they can be interpreted as

$$w_p^N W_{1+N-2p} = m_p^N \mathcal{V}_p^N = \underbrace{W_{1+N-2p} \otimes \mathcal{V}_p^N}_{} \subset \mathcal{V}$$

where A acts on first component (identity on second) and $TL_N^{(\upsilon)}$ acts on second comp. (identity on first)

<u>Remark</u>: Since A and $TL_N^{(\upsilon)}$ act on different tensor components they commute with each other as they should.

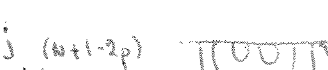
<u>Explicit basis</u> for $W_{1+N-2p} \otimes \mathcal{V}_p^N$:

$\left(F^j . w_0^{(1+N-2p)} , \right.$  $\left. \right)$ corresponds to the tensor product of

- "singlet" s in all paired points
- the state $F^j . (w_0^{\otimes(N-2p)})$ in the summand $W_{1+N-2p} \subset W_2^{\otimes(N-2p)}$ in the non-paired points.

Sketch of proof that both actions in this basis are correct:

1°) The action of A:

Consider a basis vector $\left(F^{j}_{w_0}{}^{(N+1-2p)}, \right.$  $\left. \right)$.

- When $p=0$ clearly A acts correctly on $W_2^{\otimes N}$, including the subspace $W_{N+1} \subset W_2^{\otimes N}$.
- When $p>0$ some pairs are in "singlet" and we show that if the points $l<r$ form a singlet, the A-action on the rest coincides with the action on $W_2^{\otimes(N-2)}$. Then do induction on p.

So for $p>0$ and a paired (l, r) the basis vector is a linear combination of vectors of the form

$$V = q\, V_{\alpha_1} \otimes \cdots \otimes w_{\alpha_{l-1}} \otimes w_0 \otimes w_{\alpha_{l+1}} \otimes \cdots \otimes w_{\alpha_{r-1}} \otimes w_1 \otimes w_{\alpha_{r+1}} \otimes \cdots \otimes w_{\alpha}$$
$$- V_{\alpha_1} \otimes \cdots \otimes w_{\alpha_{l-1}} \otimes w_1 \otimes w_{\alpha_{l+1}} \otimes \cdots \otimes w_{\alpha_{r-1}} \otimes w_1 \otimes w_{\alpha_{r+1}} \otimes \cdots \otimes w_{\alpha}$$

and furthermore there is an even number of points between l and r, which must be paired among themselves, so $(r-l-1)/2$ is the number of pairs "inside" (l,r). In particular we may assume $\sum_{i=l+1}^{r-1} \alpha_i = \frac{r-l-1}{2}$.

* The action of K is clearly correct since multiple coproducts are $\Delta^{(N)}(K) = K \otimes K \otimes \cdots \otimes K \otimes K$

* The action of E is given by the multiple coproduct $\Delta^{(N)}(E) = \sum_{k=0}^{N-1} \mathbb{1}^{\otimes k} \otimes E \otimes K^{\otimes(N-k-2)}$

The terms with $k \in \{0,1,2,\ldots,l-2,l,l+1,\ldots,r-2,r,\ldots,N\}$ corresponds to the action of E on $(N-2)$-fold tensor product. The remaining terms are

$$q \cdot w_{\alpha_1} \otimes \cdots \otimes \underbrace{(Ew_0)}_{=0} \otimes \cdots \otimes (Kw_1) \otimes \cdots \otimes w_{\alpha_N} \cdot q^{\sum_{j=l+1}^{r-1}(1-2\alpha_j) + \sum_{j=r+1}^{N}(1-2\alpha_j)} \to 0$$
$$+ q \cdot w_{\alpha_1} \otimes \cdots \otimes w_0 \otimes \cdots \otimes \underbrace{(Ew_1)}_{=w_0} \otimes \cdots \otimes w_{\alpha_N} \cdot q^{\sum_{j=r+1}^{N}(1-2\alpha_j)}$$
$$- w_{\alpha_1} \otimes \cdots \otimes \underbrace{(Ew_1)}_{=w_0} \otimes \cdots \otimes \underbrace{(Kw_1)}_{=qw_0} \otimes \cdots \otimes w_{\alpha_N} \cdot q^{\sum_{j=l+1}^{r-1}(1-2\alpha_j) + \sum_{j=r+1}^{N}(1-2\alpha_j)}$$
$$- w_{\alpha_1} \otimes \cdots \otimes w_1 \otimes \cdots \otimes \underbrace{Ew_0}_{=0} \otimes \cdots \otimes w_{\alpha_N} \cdot q^{\sum_{j=r+1}^{N}(1-2\alpha_j)} \to 0$$

$= 0$ so the action is OK $\left(\text{we used } \sum_{i=l+1}^{r-1}(1-2\alpha_i) = 0 \right)$

2°) The action of $TL_N^{(\nu)}$:

Consider a basis vector $\left(F^j \cdot w_0^{(1+N-2p)} , \quad \right)$ and the action of e_j on it.

There are several cases:

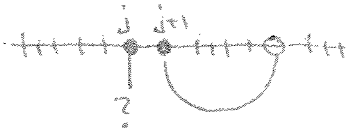
- "Both j and $j+1$ are unpaired"
  - Correct TL action is zero, a pair would be added
  - Projection to singlet vanishes, because in unpaired points we take the highest dimensional component

  $\Longrightarrow$ action of e_j is correct (zero)

- "Points j and $j+1$ are paired together"
  - Correct TL action is multiplication by ν, (a loop created)
  - projection to singlet is one, so $e_j = \nu \cdot \pi_j$ acts as ν

  $\Longrightarrow$ action of e_j is correct (mult. by ν)

- "At least one of j and $j+1$ is paired elsewhere"



The calculation essentially reduces to the action of e_1 on

$$\underbrace{q \cdot w_\alpha \otimes w_0 \otimes w_1 - w_\alpha \otimes w_1 \otimes w_0}_{= \, w_\alpha \otimes s} , \qquad \alpha \in \{0,1\}$$

and the result should be $s \otimes w_\alpha$.

Indeed

$$q \cdot w_0 \otimes w_0 \otimes w_1 - w_0 \otimes w_1 \otimes w_0 \xmapsto{e_1} 0 - \left(\nu \frac{-1}{\nu}\right) s \otimes w_0 = s \otimes w_0$$

$$q \cdot w_1 \otimes w_0 \otimes w_1 - w_1 \otimes w_1 \otimes w_0 \xmapsto{e_1} q \cdot \left(\nu \frac{q^{-1}}{\nu}\right) s \otimes w_1 - 0 = s \otimes w_1$$

$\square$