Elliptic boundary value problems with complex L^∞ coefficients and fractional regularity data

(the first order approach)

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- not assumed to be real, symmetric, or smooth in any way.

Boundary value problems

For $\theta \in [-1, 0)$ and $p > \frac{n}{n+1+\theta}$, formulate the *Regularity problem*

$$(R)^p_{\theta,A} : \begin{cases} \operatorname{div} A\nabla u = 0 \quad \text{in } \mathbb{R}^{1+n}_+, \\ ||\nabla u||_{T^p_\theta} \lesssim ||\nabla_\| f||_{\dot{H}^p_\theta}, \\ \lim_{t\to\infty} \nabla_\| u(t,\cdot) = 0 \quad \text{in } \mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^n), \\ \lim_{t\to 0} \nabla_\| u(t,\cdot) = \nabla_\| f \in \dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^n). \end{cases}$$

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Such a problem is *well-posed* if for every boundary data there *exists* a *unique* solution u satisfying the given conditions.

Notation, as promised

T^p_θ is a *(weighted) tent space*, with norm

$$||F||_{T^p_\theta} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{B(x,t)} |t^{-\theta} F(t,y)|^2 \, \frac{dy\,dt}{t^{n+1}} \right)^{p/2} dx \right)^{1/p} .$$

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$\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^n)$: $\mathbb{C}^n$-valued tempered distributions modulo polynomials.

Sketch of the first-order approach

- Identify the second-order equation $\operatorname{div} A\nabla u = 0$ with a first-order 'Cauchy–Riemann system' $\partial_t F + DBF = 0$ for

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- Note that $(F_0)_{\perp}$ and $(F_0)_{\parallel}$ are the boundary data for Regularity/Neumann problems
- Well-posedness then becomes a question of 'completing' boundary data: Given $\nabla_{\parallel} f \in \dot{H}^p_{\theta}$, can we find F_0 with $(F_0)_{\parallel} = \nabla_{\parallel} f$? Is such an F_0 unique?

Dirac operators and Cauchy–Riemann systems

For $F\colon \mathbb{R}^{1+n}_+ \to \mathbb{C}^{1+n}$,

$$DF = \begin{bmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} F_\perp \\ F_\parallel \end{bmatrix} = \begin{bmatrix} \operatorname{div} F_\parallel \\ -\nabla F_\perp \end{bmatrix}.$$

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Let $B \in L^\infty(\mathbb{R}^n : \mathcal{L}(\mathbb{C}^{1+n}))$ satisfy the same assumptions as A, and define DB as an unbounded operator on $L^2(\mathbb{R}^n : \mathbb{C}^{1+n})$.

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Associated with DB we have a *Cauchy–Riemann system*:

$$(\mathrm{CR})_{DB} : \begin{cases} \partial_t F + DBF = 0 & \text{in } \mathbb{R}^{1+n}_+, \\ F_\parallel \in \overline{\mathcal{R}(D)}. \end{cases}$$

CR-systems vs. elliptic equations

Theorem (Auscher–Axelsson–McIntosh 2010)

If u solves $\operatorname{div} A\nabla u = 0$, then the conormal gradient $\nabla_A u$ solves the Cauchy–Riemann system $(\mathrm{CR})_{D\hat{A}}$.

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In the splitting $\mathbb{C}^{1+n} = \mathbb{C} \oplus \mathbb{C}^n$,

$$A = \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} \\ A_{\|\perp} & A_{\|\|} \end{bmatrix}, \quad \hat{A} := \begin{bmatrix} I & 0 \\ A_{\|\perp} & A_{\|\|} \end{bmatrix} \begin{bmatrix} A_{\perp\perp} & A_{\perp\|} \\ 0 & I \end{bmatrix}^{-1}.$$

DB-adapted Hardy–Sobolev spaces

DB is bisectorial and has bounded H^∞ functional calculus on $\overline{\mathcal{R}(DB)} = \overline{\mathcal{R}(D)} \subset L^2$ (Auscher–Axelsson–McIntosh 2010).

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$$(\mathbb{Q}_{\varphi,DB} f)(t,x) = (\varphi(tDB)f)(x) \qquad (t > 0, f \in \overline{\mathcal{R}(DB)}).$$

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Our DB-adapted Hardy–Sobolev spaces $\mathbf{H}^p_{\theta,DB}$ are **formally** defined by

$$||f||_{\mathbf{H}^p_{\theta,DB}} := ||\mathbb{Q}_{\varphi,DB} f||_{T^p_\theta} \,,$$

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and to a (quantified) extent these spaces are independent of the choice of φ.

Spectral subspaces and Cauchy extensions

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Some particularly useful operators can be constructed:

- The *spectral projections* $\chi^+(DB)$ and $\chi^-(DB)$ defined via

  $$\chi^+(z) := \mathbf{1}_{z:\mathrm{Re}(z)>0}, \quad \chi^-(z) := \mathbf{1}_{z:\mathrm{Re}(z)<0},$$

  which induce a decomposition

  $$\mathbf{H}^p_{\theta,DB} = \mathbf{H}^{p,+}_{\theta,DB} \oplus \mathbf{H}^{p,-}_{\theta,DB}.$$

- The *Cauchy extension*

  $$\mathbf{C}_{DB} f(t) := e^{-tDB} \chi^+(DB) f \qquad (t > 0)$$

  which acts as a strongly continuous semigroup on $\mathbf{H}^{p,+}_{\theta,DB}$.

The endpoints $\theta \in \{-1, 0\}$

Theorem (Auscher–Mourgoglou 2015, Auscher–Stahlhut 2016)

Let $p > \frac{n}{n+1}$ *be such that* $\mathbf{H}^p_{0,DB} \simeq \mathbf{H}^p_{0,D}$.

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$$\left\| \widetilde{N}_*(\mathbf{C}_{DB} f) \right\|_p \simeq \|f\|_{L^p} \quad \textit{(resp. } \|\mathbf{C}_{DB} f\|_{T^{p'}_{-1}} \simeq \|f\|_{\dot{H}^{p'}_{-1}}\textit{)}, \quad \textit{and}$$

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Furthermore, $\lim_{t \to \infty} \mathbf{C}_{DB} f(t)_{\|} = 0$ *in* $\mathcal{Z}'(\mathbb{R}^n : \mathbb{C}^n)$.

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(when $p \leq 1$ we need to adjust the function spaces).

The intermediate range, $\theta \in (-1, 0)$

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Let $\theta \in (-1, 0)$ *and* $p > \frac{n}{n+1+\theta}$ *be such that* $\mathbf{H}^p_{\theta,DB} \simeq \mathbf{H}^p_{\theta,D}$.

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This does *not* follow from the previous theorem by interpolation. Our proof only covers $\theta \in (-1, 0)$: it does not recover the previous theorem.

Classification of well-posedness

When $\theta \in [-1, 0]$ and $p > \frac{n}{n+1+\theta}$,

$$\begin{aligned} \mathbf{H}^p_{\theta,D} &= \dot{H}^p_\theta(\mathbb{R}^n) \oplus (\dot{H}^p_\theta(\mathbb{R}^n : \mathbb{C}^n) \cap \mathcal{N}(\mathrm{curl})) \\ &=: \mathbf{H}^p_{\theta,\perp} \oplus \mathbf{H}^p_{\theta,\|}. \end{aligned}$$

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Let $N_\perp$ and $N_\parallel$ denote the associated projections on $\mathbf{H}^p_{\theta,D}$.
For (p,θ) such that $\mathbf{H}^p_{\theta,DB} \simeq \mathbf{H}^p_{\theta,D}$, we can identify $\mathbf{H}^{p,+}_{\theta,DB} \subset \mathbf{H}^p_{\theta,D}$, and then restrict the projections $N_\perp$ and $N_\parallel$ to define

$$N^{(p,\theta)}_{DB,\perp} : \mathbf{H}^{p,+}_{\theta,DB} \to \mathbf{H}^p_{\theta,\perp} \quad \text{and} \quad N^{(p,\theta)}_{DB,\parallel} : \mathbf{H}^{p,+}_{\theta,DB} \to \mathbf{H}^p_{\theta,\parallel}.$$

Theorem (Auscher–Mourgoglou 2014, AA–Auscher 2016)

Let $\theta \in [-1,0]$, and $p > \frac{n}{n+1+\theta}$. Suppose that $\mathbf{H}^p_{\theta,D\hat{A}} = \mathbf{H}^p_{\theta,D}$. Then $(R)^p_{\theta,A}$ (resp. $(N)^p_{\theta,A}$) is well-posed if and only if $N^{(p,\theta)}_{D\hat{A},\parallel}$ (resp. $N^{(p,\theta)}_{D\hat{A},\perp}$) is an isomorphism.

Consequences

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- Some stability in coefficients: w-p of $(R)^p_{\theta,A}$ implies w-p of $(R)^p_{\theta,\tilde{A}}$ for $\|\tilde{A}-A\|_\infty$ sufficiently small (with some restrictions on (p,θ)).

What about Besov spaces?

We can Hardy–Sobolev spaces $\dot{H}^p_\theta$ with Besov spaces $\dot{B}^{p,p}_\theta$, and tent spaces T^p_θ with the spaces Z^p_θ defined by

$$||F||_{Z^p_\theta} := \left(\iint_{\mathbb{R}^{1+n}_+} \left(⨍_{t/2}^{2t} ⨍_{B(x,t)} |\tau^{-\theta} F(\tau,\xi)|^2 \, d\xi \, d\tau \right)^{p/2} dx \, \frac{dt}{t} \right)^{1/p} .$$

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Note that

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One can even use interpolation to deduce well-posedness of BVPs with data in $\dot{B}^{p,p}_\theta$ from endpoint problems with data in $\dot{H}^p_\theta$. Thus we recover some of the results of Barton and Mayboroda (2016).

Thanks for your attention!