The maximal operator, weights and extrapolation on variable Lebesgue spaces

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Joint work with (a cast of thousands):

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Intuition

Classical Lebesgue spaces:

$$\int_\Omega |f(x)|^p \, dx < \infty, \qquad 1 \leq p < \infty.$$

Variable Lebesgue spaces:

$$\int_\Omega |f(x)|^{p(x)} \, dx < \infty.$$

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Motivation: Function Spaces, PDEs and variational integrals

Example of a Musielak-Orlicz space

Minimization problem:

$$\mathcal{F}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\,dx$$

Euler-Lagrange equation: $p(\cdot)$-Laplacian

$$\operatorname{div}\big(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u\big) = 0$$

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Applications

Electrorheological fluids: space dependent energy estimates

Image restoration: L^1 estimates at boundaries, L^2 estimates in interior

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A meta-theorem

If a theorem is true in L^p, it is probably true in $L^{p(\cdot)}$,
but proving it will be an adventure.

Exponent functions

$$p(\cdot) \in \mathcal{P}(\Omega) \qquad p(\cdot) : \Omega \to [1, \infty]$$

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$$

For $E \subset \Omega$

$$p_-(E) = \operatorname{ess\,inf}\{p(x) : x \in E\}$$
$$p_+(E) = \operatorname{ess\,sup}\{p(x) : x \in E\}$$

Hereafter: $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$

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The modular & norm

Given $p(\cdot) \in \mathcal{P}(\Omega)$ define the modular:

$$\rho_{p(\cdot)}(f) = \rho(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)}$$

and the Luxemburg norm:

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \le 1\right\}$$

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The space $L^{p(\cdot)}$

Theorem

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $\|\cdot\|_{p(\cdot)}$ is a norm and

$$L^{p(\cdot)}(\Omega) = \{f : \|f\|_{p(\cdot)} < \infty\}$$

is a Banach function space.

The maximal operator

Given $f \in L^1_{loc}(\mathbb{R}^n)$, define

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy = \sup_{Q \ni x} ⨍_Q |f(y)|\,dy.$$

Theorem

Given $1 < p \leq \infty$,

$$\|Mf\|_p \leq C\|f\|_p.$$

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Local log-Hölder continuity

Given $p(\cdot)$, we say $1/p(\cdot) \in LH_0$ if

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{C_0}{-\log(|x-y|)}, \qquad |x-y| < \frac{1}{2}$$

If $p_+ < \infty$, $p(\cdot) \in LH_0 \Leftrightarrow 1/p(\cdot) \in LH_0$.

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Log-Hölder continuity at infinity

Given $p(\cdot)$, we say $1/p(\cdot) \in LH_\infty$ if there exists p_∞ such that

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{C_\infty}{\log(e + |x|)}.$$

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Hereafter, let $LH = LH_0 \cap LH_\infty$.

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$L^{p(\cdot)}$ estimates for the maximal operator

Theorem (Diening; Nekvinda; DCU,AF,CJN)

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ be such that $1/p(\cdot) \in LH$ and $p_- > 1$; then

$$\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

LH_0 and LH_∞ pointwise sharp

Take any $\phi(\cdot) : [0,\infty) \to [0,1]$, increasing, smooth $\phi(x) = 0$ and

$$\lim_{x\to 0^+} \phi(x)\log(x) = -\infty.$$

Define

$$p(x) = \begin{cases} 2+\phi(x) & x \geq 0 \\ 2 & x < 0 \end{cases}$$

Then $p(\cdot) \notin LH_0$ and M not bounded on $L^{p(\cdot)}(\mathbb{R})$.

Similar construction holds for LH_∞.

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Continuity not necessary

Theorem (Lerner)

Let

$$p(x) = 2 + \alpha \sin(\pi \log\log(|x| + 1/|x|));$$

Then $p(\cdot)$ is not continuous at 0 or infinity

But for all $\alpha > 0$ small, M is bounded on $L^{p(\cdot)}(\mathbb{R})$.

A_p weights

A non-negative, locally integrable function w is in A_p, $1 < p < \infty$, if

$$[w]_{A_p} = \sup_Q \fint_Q w(x)\,dx \left(\fint_Q w(x)^{1-p'}\,dx \right)^{p-1} < \infty.$$

Classical inequalities

Theorem

For $1 < p < \infty$ *TFAE*

- $w \in A_p$
- $$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx$$
- $$\int_{\mathbb{R}^n} |R_j f(x)|^p w(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx, \quad 1 \leq j \leq n$$

Weights as multipliers

Replace weight w by w^p, and restate A_p condition as

$$\sup_Q |Q|^{-1} \|w\chi_Q\|_p \|w^{-1}\chi_Q\|_{p'} < \infty, \quad 1 \leq p \leq \infty.$$

Then for $1 < p \leq \infty$ restate norm inequalities as

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Variable A_p weights

Given an exponent $p(\cdot)$, a weight $w \in A_{p(\cdot)}$ if

$$[w]_{A_{p(\cdot)}} = \sup_Q |Q|^{-1} \|w\chi_Q\|_{p(\cdot)} \|w^{-1}\chi_Q\|_{p'(\cdot)} < \infty.$$

Main result

Theorem (LD,PH; DCU, LD, PH; DCU,AF,CJN)

Given $p(\cdot)$, $1 < p_- \leq p_+ < \infty$, *such that* $p(\cdot) \in LH$, *if* $w \in A_{p(\cdot)}$,

$$\|(Mf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Conversely, if the weighted inequality holds, then $w \in A_{p(\cdot)}$ *(even without* $p(\cdot) \in LH$*).*

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Problem with unweighted proof

Unweighted proof used norm inequalities for maximal operator on L^{p_-} and L^{p_∞}

This would require $w(\cdot)^{p(\cdot)} \in A_{p_-}$

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A key lemma

Lemma

If $p(\cdot) \in LH$ and $w \in A_{p(\cdot)}$, then $w(\cdot)^{p(\cdot)} \in A_\infty$ (A_q, $q > p_+$)

Alternate proof of Christ and Fefferman

- Discretize:

$$\int_{\mathbb{R}^n} Mf(x)^p w(x)\,dx \approx \sum_{k,j} \left(\fint_{Q_j^k} |f(y)|\,dy \right)^p w(E_j^k)$$

- Definition of A_p: let $\sigma = w^{1-p'}$

$$\sum_{k,j} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} |f(y)|\sigma^{-1}\sigma\,dy \right)^p \sigma(Q_j^k) \frac{w(E_j^k)}{|Q_j^k|} \left(\frac{\sigma(Q_j^k)}{|Q_j^k|} \right)^{p-1}$$

- Weighted maximal operator

$$\sum_{k,j} \int_{E_j^k} M_\sigma(f\sigma^{-1})(x)^p \sigma(x)\,dx$$

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Adapting this proof to weighted $L^{p(\cdot)}$

- $f \geq 0$, bounded, compact support, $\|fw\|_{p(\cdot)} = 1$
- $f = f_1 + f_2$, $f_1 = f\chi_{\{f\sigma^{-1}>1\}}$, $f_2 = f\chi_{\{f\sigma^{-1}\leq 1\}}$, $\sigma = w(\cdot)^{-p'(\cdot)}$
- estimate f_1 using LH_0, f_2 using LH_∞
- Discretize both terms

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Estimate for f_1

- Hölder's inequality "works":

$$\int Mf_1(x)^{p(x)} w(x)^{p(x)}\,dx$$
$$\lesssim \sum_{j,k} \left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} [f_1(y)\sigma(y)^{-1}]^{p(y)/p_-} \sigma(y)\,dy \right)^{p_-}$$
$$\times \int_{Q_j^k} \sigma(Q_j^k)^{p_-(Q_j^k)} |Q_j^k|^{-p(x)} w(x)^{p(x)}\,dx$$

- Use $A_{p(\cdot)}$ and A_∞ to bound last term by $\sigma(Q_j^k) \lesssim \sigma(E_j^k)$
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Estimate for f_2

- Multiple cases depending on size of cubes
- Simplest case

$$\sum \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\,dy \right)^{p(x)} w(x)^{p(x)}\,dx$$
$$\lesssim \sum \int_{E_j^k} \left(\fint_{Q_j^k} f_2(y)\,dy \right)^{p_\infty} w(x)^{p(x)}\,dx$$
$$+ \sum \int_{E_j^k} \frac{w(x)^{p(x)}}{(e+|x|)^{ntp_-}}\,dx$$

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Five open questions

- Is theorem true when $p_+ = \infty$?
- Can we eliminate LH condition?

  Conjecture [LD,PH]: M bounded on $L^{p(\cdot)}$ and $w \in A_{p(\cdot)}$
- Generalize to fractional, one-sided maximal operators
- Generalize to spaces of homogeneous type
- Find the sharp constants in terms of $[w]_{A_{p(\cdot)}}$

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Extrapolation in $L^{p(\cdot)}$

Theorem (DCU, DW)

Given an operator T, suppose that for some p_0, $1 \leq p_0 < \infty$, and all $w \in A_{p_0}$,

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x)\, dx \leq C([w]_{A_{p_0}}) \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x)\, dx.$$

Let $p(\cdot)$ be such that $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH$. Then for $w \in A_{p(\cdot)}$,

$$\|(Tf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Similar results hold for off-diagonal ($\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \gamma$) and limited range ($q_- < p_- \leq p_+ < q_+$) extrapolation.

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Sketch of proof

- Follow proof of RdF extrapolation due to DCU, JMM, CP.
- Deal very carefully with choice of duality exponent.
- Use fact that M is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$.
- Lerner showed that if $w^{p(\cdot)} \in A_\infty$, the second follows from first w/o assuming $p(\cdot) \in LH$.

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Singular integrals in $L^{p(\cdot)}$

Corollary (DCU, DW)

If T is a Calderón-Zygmund singular integral, given $p(\cdot) \in LH$ such that $1 < p_- < p_+ < \infty$, then for $w \in A_{p(\cdot)}$,

$$\|(Tf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Factorization

Theorem (DCU,DW)

Given $w_1,\ w_2 \in A_1$ and $p(\cdot) \in LH$, $1 < p_- \leq p_+ < \infty$, then

$$w^{1/p(\cdot)} w^{-1/p'(\cdot)} \in A_{p(\cdot)} \cap A_2.$$

Theorem (DCU, DW)

$$A_\infty = \bigcup A_{p(\cdot)},$$

where union is taken over all $p(\cdot) \in LH$, $1 < p_- \leq p_+ < \infty$.

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Kiitos Paljon!

Roll Tide!