

# News on Weights

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May 2016

In this talk, we present several new directions in weighted theory, generalising existing classical results on CZOs.

- Non-kernel operators
- Martingales, change of law
- What time permits

# Origin, non-kernel operators

With  $A$  an accretive ( $\operatorname{Re}(A(x)) \geq \kappa I$ ) matrix with  $L^\infty$  coefficients and  $L = -\operatorname{div}(A\nabla)$ , define the Riesz transforms

$$\nabla L^{-1/2}$$

There may be only a range of  $p_0 < 2 < q_0$  of boundedness (even without any weight) determined in part by the limitations in range of  $L^p$  estimates for the semigroup

$$(e^{-tL})_t$$

# Semigroup

When pointwise heat kernel bounds fail:

$B_1$  and  $B_2$  of scale  $\sqrt{t}$

$$\|e^{-tL}\|_{L^{p_0}(B_1) \rightarrow L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0} |B_2|^{1/q_0} e^{-c \frac{d(B_1, B_2)^2}{t}}$$

Here,  $L$  is linear, injective,  $\theta$  accretive densely defined in  $L^2$ . Under these assumptions  $-L$  generates an analytic semigroup in  $L^2$  (one that extends in  $t$  to an angle where it is analytic in an appropriate sense).

# Littlewood Paley operators

Derive operators  $P_t$  and  $Q_t$  from this semigroup:

$P_t$  role of approximate identity at scale  $\sqrt{t}$

$Q_t$  cancellative

For  $N > 0$  define

$$Q_t^{(N)} \sim (tL)^N e^{-tL} \text{ with } Q_t^{(1)} = tLe^{-tL}$$

$$P_t^{(N)} = \int_0^\infty Q_{st}^{(N)} \frac{ds}{s} \text{ with } P_t^{(1)} = e^{-tL}$$

$$t\partial_t P_t^{(N)} = -Q_t^{(N)}$$

# Littlewood Paley operators

$P_t$  and  $Q_t$  are uniformly in  $t$  bounded in  $L^p$  (in the  $p_0, q_0$  range) with off diagonal estimates at scale  $\sqrt{t}$ .

$$\lim_{t \rightarrow 0} P_t^{(N)} f = f, \quad \lim_{t \rightarrow \infty} P_t^{(N)} f = 0 \quad (L^p)$$

$$Id = \int_0^\infty Q_t^{(N)} \frac{dt}{t} \quad (L^p)$$

$$P_t^{(N)} = Id - \int_0^t Q_s^{(N)} \frac{ds}{s}$$

# Operators $T$

We work with an axiomatic setting for operators  $T$  that will include many interesting examples.

- $T$  is bounded in  $L^2$
- $T_t = TQ_t^{(N)}$  has off-diagonal estimates at scale  $\sqrt{t}$

$$\|T_t\|_{L^{p_0}(B_1) \rightarrow L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0} |B_2|^{1/q_0} \left(1 + \frac{d(B_1, B_2)^2}{t}\right)^{-(\nu+1)/2}$$

- condition serving for off-diagonal estimates for low frequencies:

$$\langle |Te^{-r^2 L} f|^{q_0} \rangle_{B(x,r)}^{1/q_0} \lesssim \inf_{y \in B(x,r)} M_{p_1}(Tf)(y) + \inf_{y \in B(x,r)} M_{p_1}(f)(y)$$

some  $p_0 \leq p_1 < 2$  all  $r > 0$

# Example $T$

Holomorphic functional calculus of  $L$ .

Dirichlet Forms.

Riesz transforms (divergence type, riemannian manifold non-negative curvature)

Paraproducts

Fourier multipliers



## Associated Maximal operators

$$\begin{aligned} T_{q_0}^\sharp f &= \sup_{x \in B} \langle |T \int_{r(B)^2}^\infty Q_t^{(N)} f \frac{dt}{t} |^{q_0} \rangle_B^{1/q_0} \\ &= \sup_{x \in B} \langle |TP_{r(B)^2}^{(N)} f|^{q_0} \rangle_B^{1/q_0} \end{aligned}$$

There holds  $|Tf| \leq T^\sharp f$  almost everywhere.

$T^\sharp$  is of weak type  $p_0$  and bounded in  $L^p$  for  $p_0 < p \leq 2$ .

$$M_{p_0}^* f(x) = \sup_{x \in Q} \inf_{y \in Q} (M|f|^{p_0})^{1/p_0}(y)$$

is almost everywhere just  $M_{p_0} f$ .

## Localised associated Maximal operators

$$T_{Q_0, q_0}^\sharp f(x) = \sup_{x \in Q \subset Q_0} \langle |T \int_{r(Q)^2}^{\infty} Q_t^{(N)} f \frac{dt}{t}|^{q_0} \rangle_Q^{1/q_0}$$

$$M_{Q_0, p_0}^* f(x) = \sup_{x \in Q \subset Q_0} \inf_{y \in Q} (M|f|^{p_0})^{1/p_0}(y)$$

# Sparse decomposition

With the set

$$E = \{x \in Q_0 : \max(T_{Q_0, q_0}^\# f(x), M_{Q_0, p_0}^* f(x)) > \eta \langle |f|^{p_0} \rangle_{5Q_0}^{1/p_0}\}$$

we eventually get the first step in the sparse decomposition

$$\left| \int_{Q_0} T f \cdot g \right| \lesssim \sum_{P \in \mathcal{S}} \mu(P) \langle |f|^{p_0} \rangle_{5P}^{1/p_0} \langle |f|^{q'_0} \rangle_{5P}^{1/q'_0}$$

# Weight characteristics

$$[W]_{A_p} = \sup_B \langle W \rangle_B \langle W^{1-p'} \rangle_B^{p-1}$$

$$[W]_{RH_q} = \sup_B \langle W^q \rangle_B^{1/q} \langle W \rangle_B^{-1}$$

$$[W]_{p_0, q_0, p} = [W]_{A_{\frac{p}{p_0}}} [W]_{RH_{\frac{q_0}{p}}}$$

## Weight characteristics, critical power weights

$n = 1$  and  $w_a = |x|^a$  when  $\epsilon \rightarrow 0$

$$[w_{-1+\epsilon}]_{A_p} \sim \frac{1}{\epsilon}$$

$$[w_{p-1-\epsilon}]_{A_p} \sim \frac{1}{\epsilon^{p-1}}$$

$$[w_{-1/s+\epsilon}]_{RH_s} \sim \frac{1}{\epsilon^{1/s}}$$

# Weighted estimate

There holds

$$\|S\|_{L^p(w) \rightarrow L^p(w)} \lesssim [w]_{p_0, q_0, p}^\alpha$$

with

$$\alpha = \max\left(\frac{1}{p - p_0}, \frac{q_0 - 1}{q_0 - p}\right)$$

dualizing through  $\bar{p} = 1 + \frac{p_0}{q'}$

These are the usual exponents if  $p_0 = 1$  and  $q_0 = \infty$ .

# Wittwer's theorem

Recall the mother of modern weighted theory, on the probability space  $[0, 1]$  endowed with Lebesgue measure and  $h_I$  the Haar system.

$$T_\sigma : h_I \mapsto \sigma_I h_I$$

A theorem by Wittwer states that uniformly in  $|\sigma|_\infty \leq 1$ ,

$$\|T_\sigma\|_{L^2(w) \rightarrow L^2(w)} \lesssim Q_2(w)$$

where dyadic  $A_2$  stands.

# Probability for Analysts 1

$[0, 1]$  with Lebesgue measure is a probability space.

For each discrete time  $0 \leq n$  consider the classical dyadic covering of size  $2^{-n}$ , together with its generated sigma algebra.

This becomes a filtered probability space  $([0, 1], F, dx)$

$$X_n = E(f|F_n) \text{ and } Y_n = E(T_\sigma f|F_n)$$

are a pair of martingales that are differentially subordinate.

Indeed,  $E(f|F_n) = \sum_{1 \leq k \leq n} \Delta_k(f) + E(f)$ . So

$$|\Delta_n(T_\sigma f)| \leq |\sigma_n| |\Delta_n(f)|$$

Here,  $dX_n = \Delta_n(f) = E(f|F_n) - E(f|F_{n-1})$

Note  $\sigma_n$  is measurable in  $F_n$ , such multipliers are called predictable.



# TTV result

Predictable multipliers in general probability spaces with discrete in time filtrations have sharp weighted estimates using the martingale  $A_2$  characteristic.

# Probability for Analysts 2

Square bracket, discrete filtration:

$$[X, X]_n = \sum_{k=1}^n (dX_k)^2$$

for example discrete random walk  $B$ :  $[B, B]_n = \sum_{k=1}^n 1 = n$

Modern probability theory is concerned with filtered probability spaces with continuous time:  $(\Omega, F, \mu)$ , for example the Brownian filtration.

# Probability for Analysts 3

Square bracket and products (almost surely):

$$(XX)_n - (XX)_{n-1} = 2X_{n-1}(X_n - X_{n-1}) + (X_n - X_{n-1})^2 = 2X_{n-1}dX_n + (dX_n)^2$$

This can be generalised to continuous in time filtrations and defined the square bracket as the predictable compensator of the product of martingales. One obtains the bracket process:

$$[X, X] = X^2 - \int X_- dX$$

Indeed,  $X_0^2 + \sum_i (X^{T_{i+1}^n} - X^{T_i^n})$  with  $T^n$  for all  $n$  sequence of increasing stopping times.

# Differential subordination

$Y$  differentially subordinate to  $X$  if  $[X, X]_t - [Y, Y]_t$  is a non-negative and non-decreasing function of  $t \geq 0$ .

If the martingale has discontinuous paths (jumps) then this bracket differs from  $\langle \cdot, \cdot \rangle$  in that subordination gives precise information at the instances of jumps.

# Theorem and proof

## Theorem (Domelevo, P.)

*$Y$  differentially subordinate to  $X$  (with values in Hilbert space) then sharp weighted estimates hold using the martingale characteristic of the weight.*

Bellman function of four variables for the entire problem in the weak form with

- not quite enough regularity
- need  $\tau$  lemma with control on  $\tau$  (not needed for simpler cases)
- precise discrete one-leg convexity of its pieces gives global discrete concavity in non-convex domain (needed in presence of jumps)
- stopping jump process
- cutting weight and dimension, an alphabet worth of limiting procedures

# Why is this related to CZO?

There is the famous formula of Gundy Varopoulos. There is also one for jumps (second order Riesz transforms) with strong subordination

Theorem (Arcozzi, Domelevo, P. (2015))

*The second order Riesz transforms  $R_i^2 f$ ,  $1 \leq i \leq m$ , and  $R_{jk} f$ ,  $1 \leq j, k \leq n$ , of a function  $f \in L^2(G)$  as defined in can be written as the conditional expectations*

$$R_i^2 f(z) = \mathbb{E}(M^{i,f} | \mathcal{Z}_0 = z) \text{ and } R_{jk} f(z) = \mathbb{E}(M^{j,k,f} | \mathcal{Z}_0 = z).$$

*Here  $M_t^{i,f}$  and  $M_t^{j,k,f}$  are suitable martingale transforms of the martingale  $M_t^f$  associated to  $f$ , and  $\mathcal{Z}_t$  is a suitable random walk on  $G$*

# Other news

Kamilia Dahmani is finishing her project on a weighted estimate for Riesz transforms on manifolds of nonnegative Bakry-Emery-curvature, endowed with measure  $e^{-\phi}$ . This is new even in the the case of the Gauss space.