Norm-variation of bilinear averages

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joint work with

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Single averages

$$M_n f(x) := \frac{1}{n} \sum_{i=0}^{n-1} f(S^i x)$$

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The setting:

- $(X, \mathcal{F}, \mu)$ a probability space
- $S\colon X \to X$ measure-preserving: $\mu\big(S^{-1}E\big) = \mu(E)$
- $f \in \mathrm{L}^2(X)$

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Convergence as $n \to \infty$:

- in $\mathrm{L}^2(X)$ $\rightsquigarrow$ von Neumann (1932)
- pointwise a.e. $\rightsquigarrow$ Birkhoff (1931)
- Can we quantify the L^2 convergence by controlling the number of jumps in the norm?

Single averages

$$M_n f(x) := \frac{1}{n} \sum_{i=0}^{n-1} f(S^i x)$$

Norm-variation estimate [Jones, Ostrovskii, and Rosenblatt (1996)]

$$\sup_{n_0 < n_1 < \cdots < n_m} \sum_{j=1}^{m} \left\| M_{n_j} f - M_{n_{j-1}} f \right\|_{\mathrm{L}^2}^2 \le C \, \|f\|_{\mathrm{L}^2}^2$$

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Consequence

If $\|f\|_{\mathrm{L}^2} = 1$, then $\left(M_n f\right)_{n=1}^{\infty}$ has $O(\varepsilon^{-2})$ jumps of size $\geq \varepsilon$ in the L^2 norm

$m_1 < n_1 \leq m_2 < n_2 \leq \cdots \leq m_J < n_J$ s.t. $\|M_{n_j} f - M_{m_j} f\|_{\mathrm{L}^2} \geq \varepsilon$

Double averages

$$M_n(f,g)(x) := \frac{1}{n}\sum_{i=0}^{n-1} f(S^i x)g(T^i x)$$

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- $f, g \in \mathrm{L}^\infty(X)$
- motivated by Furstenberg's work in additive combinatorics

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Convergence as $n \to \infty$:

- in $\mathrm{L}^2(X)$ (and in $\mathrm{L}^p(X)$, $p < \infty$) $\rightsquigarrow$ Conze and Lesigne (1984)
- pointwise a.e. $\rightsquigarrow$ an old open problem! (Calderón?)
- quantitative L^2 convergence? metric entropy bounds?

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Previous results:

- $T = S^m$, $m \in \mathbb{Z}$, pointwise a.e. convergence
  $\rightsquigarrow$ Bourgain (1990), Demeter (2007)
- $T = S^m$, $m \in \mathbb{Z}$, pointwise variation estimate
  $\rightsquigarrow$ Do, Oberlin, and Palsson (2015)

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General commuting and measure-preserving S, T:
any explicit quantitative results for norm convergence
$\rightsquigarrow$ posed by Avigad and Rute (2012) and others

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Norm-variation estimate [Durcik, K., Škreb, and Thiele (2016)]

$$\sup_{n_0<n_1<\cdots<n_m} \sum_{j=1}^{m} \left\| M_{n_j}(f,g) - M_{n_{j-1}}(f,g) \right\|_{\mathrm{L}^2}^2 \le C \, \|f\|_{\mathrm{L}^4}^2 \|g\|_{\mathrm{L}^4}^2$$

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Consequence

If $\|f\|_{\mathrm{L}^4} = \|g\|_{\mathrm{L}^4} = 1$, then $\left(M_n(f,g)\right)_{n=1}^{\infty}$ has $O(\varepsilon^{-2})$ jumps of size $\geq \varepsilon$ in the L^2 norm

Multiple averages

$$M_n(f_1, f_2, \ldots, f_r)(x) := \frac{1}{n}\sum_{i=0}^{n-1} f_1(T_1^i x) f_2(T_2^i x) \cdots f_r(T_r^i x)$$

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The setting:

- $r \geq 3$
- $(X, \mathcal{F}, \mu)$ a probability space
- $T_1, T_2, \dots, T_r \colon X \to X$ commuting and measure-preserving
- $f_1, f_2, \dots, f_r \in \mathrm{L}^\infty(X)$

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Convergence as $n \to \infty$:

- in $\mathrm{L}^2(X)$ (and in $\mathrm{L}^p(X)$, $p < \infty$) $\rightsquigarrow$ Tao (2007) and also by Austin (2008), Walsh (2011), Zorin-Kranich (2011)
- no quantitative norm-convergence results known

Bilinear averages

$$A_t^{\varphi}(F,G)(x,y) := \int_{\mathbb{R}} F(x+s,y)G(x,y+s)\, t^{-1}\varphi(t^{-1}s)\, ds$$

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Norm-variation estimate [DKŠT (2016)]

$$\sup_{t_0<\cdots<t_m} \sum_{j=1}^{m} \|A_{t_j}^{\varphi}(F,G) - A_{t_{j-1}}^{\varphi}(F,G)\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 \le C_{\varphi} \|F\|_{\mathrm{L}^4(\mathbb{R}^2)}^2 \|G\|_{\mathrm{L}^4(\mathbb{R}^2)}^2$$

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- φ is a Schwartz function
- $\varphi = \mathbb{1}_{[0,1)} \implies A_t(F,G)(x,y) = \frac{1}{t}\int_0^t F(x+s,y)G(x,y+s)ds$

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$t_j = 2^j$, φ Schwartz

$\psi(s) := \varphi(s) - 2\varphi(2s) \implies \psi_{2^j} = \varphi_{2^j} - \varphi_{2^{j-1}}$

$$S(F,G)(x,y) := \Big(\sum_{j\in\mathbb{Z}} \Big| \underbrace{\int_{\mathbb{R}} F(x+s,y)G(x,y+s)\psi_{2^j}(s)ds}_{A^{\varphi}_{2^j}(F,G)(x,y) - A^{\varphi}_{2^{j-1}}(F,G)(x,y)} \Big|^2\Big)^{1/2}$$

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Consequence

$\|S(F,G)\|_{L^2(\mathbb{R}^2)} \leq C_{\psi}\|F\|_{L^4(\mathbb{R}^2)}\|G\|_{L^4(\mathbb{R}^2)}$

Bilinear averages

$$A_t^{\varphi}(F,G)(x,y) := \int_{\mathbb{R}} F(x+s,y)G(x,y+s)\,\underbrace{t^{-1}\varphi(t^{-1}s)}_{\varphi_t(s)}\,ds$$

$$\widetilde{S}(f,g)(x) := \Big(\sum_{j\in\mathbb{Z}} \Big|\int_{\mathbb{R}} f(x+s)\,g(x-s)\,2^{-j}\psi(2^{-j}s)\,ds\Big|^2\Big)^{1/2}$$

$$F(x,y) := f(x-y)R^{-1/4}\vartheta(R^{-1}y)$$
$$G(x,y) := g(x-y)R^{-1/4}\vartheta(R^{-1}x)$$

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Consequence [Lacey, Thiele (1997), bilinear Hilbert transform]

$\|\widetilde{S}(f,g)\|_{L^2(\mathbb{R})} \le C_\psi \|f\|_{L^4(\mathbb{R})}\|g\|_{L^4(\mathbb{R})}$

Bilinear averages

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Compare with the "triangular Hilbert transform":

$$T(F,G)(x,y) := \text{p.v.}\int_{\mathbb{R}} F(x+s,y)\,G(x,y+s)\,\frac{ds}{s}$$

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$$T(F,G)(x,y) := \text{p.v.} \int_{\mathbb{R}} F(x+s,y)\, G(x,y+s) \frac{ds}{s}$$

- one function specialized $\rightsquigarrow$ K., Thiele, Zorin-Kranich (2015)
- nontrivial cancellation $\rightsquigarrow$ Zorin-Kranich (2015)
- no-estimates known!
- $\text{p.v.}\frac{1}{s} = \sum_j \psi_{2^j}(s) \rightsquigarrow$ the square function vs. the singular integral

The “triangular Hilbert transform”:

$$T(F,G)(x,y) := \text{p.v.} \int_{\mathbb{R}} F(x+s,y)\, G(x,y+s)\frac{ds}{s}$$

It has to be difficult! Dualize:

$$\Lambda(F,G,H) = \int_{\mathbb{R}^2} \text{p.v.} \int_{\mathbb{R}} F(x+s,y)G(x,y+s)H(x,y)\frac{ds}{s}dxdy$$

By taking

$$F(x,y) = f(x),\ G(x,y) = g(x)e^{iN(x)y},\ H(x,y) = h(x)e^{-iN(x)y}$$

we obtain bounds for the linearized Carleson operator

$$(Cf)(x) = \int_{\mathbb{R}} f(x+s)e^{iN(x)s}\frac{ds}{s}$$

$$\Lambda(F,G,H) = \langle Cf, gh \rangle$$

$\rightsquigarrow$ Carleson (1966)

Bilinear averages on $\mathbb{Z}^2$

$$\widetilde{A}_n(\widetilde{F},\widetilde{G})(k,l) := \frac{1}{n}\sum_{i=0}^{n-1}\widetilde{F}(k+i,l)\,\widetilde{G}(k,l+i)$$

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$$\sup_{n_0<\cdots<n_m}\sum_{j=1}^{m}\|A_{n_j}(\widetilde{F},\widetilde{G})-A_{n_{j-1}}(\widetilde{F},\widetilde{G})\|_{\ell^2(\mathbb{Z}^2)}^2 \le C\|\widetilde{F}\|_{\ell^4(\mathbb{Z}^2)}^2\|\widetilde{G}\|_{\ell^4(\mathbb{Z}^2)}^2$$

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Calderón's transference: $\mathbb{R}^2 \dashrightarrow \mathbb{Z}^2 \dashrightarrow (X,\mathcal{F},\mu,S,T)$

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$$F(x,y) := \sum_{i,l\in\mathbb{Z}} \widetilde{F}(i-l, l)\, \mathbb{1}_{[i,i+1)}(x+y)\, \mathbb{1}_{[l,l+1)}(y)$$

$$G(x,y) := \sum_{i,k\in\mathbb{Z}} \widetilde{G}(k, i-k)\, \mathbb{1}_{[k,k+1)}(x)\, \mathbb{1}_{[i,i+1)}(x+y)$$

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$$\widetilde{F}_x(k,l) := f(S^k T^l x)$$
$$\widetilde{G}_x(k,l) := g(S^k T^l x)$$

§3. Cheap tricks

- just the ideas, technical oversimplifications
- sometimes work in discrete, sometimes in continuous scales
- sometimes apply before, sometimes after decompositions

Split into “long” and “short” variations
[Jones, Seeger, Wright (2004)]:

$$t_0 < t_1 < \cdots t_{j-1} < t_j < \cdots < t_m$$

Insert the closest powers 2^k, $k \in \mathbb{Z}$

- Long variation / long jumps:
  corresponding to the scales $t_j \in \{2^k : k \in \mathbb{Z}\}$
- Short variation / short jumps:
  corresponding to t_j from a fixed interval $[2^k, 2^{k+1}]$

Expand out the L^2 norm for the long variation:

$$\sum_{j=1}^{m} \left\| A^{\varphi}_{2^{k_j}}(F,G) - A^{\varphi}_{2^{k_{j-1}}}(F,G) \right\|^2_{L^2(\mathbb{R}^2)}$$
$$= \sum_{j=1}^{m} \int_{\mathbb{R}^4} F(x+u,y)G(x,y+u)F(x+v,y)G(x,y+v)$$
$$(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)\, dxdydudv$$

A telescoping identity:

$$(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)$$
$$= \underbrace{\varphi_{2^{k_{j-1}}}(u)\varphi_{2^{k_{j-1}}}(v) - \varphi_{2^{k_j}}(u)\varphi_{2^{k_j}}(v)}_{\text{telescopes into a single-scale quantity}}$$
$$+ \varphi_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)$$
$$+ (\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(u)\varphi_{2^{k_j}}(v)$$

Kernel of the 4-linear form:

$$K(u,v) := \sum_{j=1}^{m} \varphi_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)$$

Denote $\psi := \varphi - \varphi_2$

$$\varphi_{2^{k_{j-1}}} - \varphi_{2^{k_j}} = \sum_{l=k_{j-1}}^{k_j-1} \psi_{2^l}$$

Leads to a single-index sum (one parameter),
not a double-index one (two parameters)

$$F(x+u,y)G(x,y+u)F(x+v,y)G(x,y+v)\,\varphi(u)\psi(v)$$

Change of variables:

$$z = x+y+u, \quad w = x+y+v$$

leads to:

$$\widetilde{F}(y,z)\widetilde{G}(x,z)\widetilde{F}(y,w)\widetilde{G}(x,w)\,\varphi(z-x-y)\psi(w-x-y)$$

The Fourier inversion formula:

$$\begin{aligned}&\varphi(z-x-y)\psi(w-x-y)\\&=\int_{\mathbb{R}^2}\widehat{\varphi}(\xi)\widehat{\psi}(\eta)e^{2\pi i\xi(z-x-y)}e^{2\pi i\eta(w-x-y)}d\xi d\eta\end{aligned}$$

$$\varphi(z-x-y)\psi(w-x-y)$$
$$= \int_{\mathbb{R}^2} \widehat{\varphi}(\xi)\widehat{\psi}(\eta)e^{-2\pi i(\xi+\eta)x}e^{-2\pi i(\xi+\eta)y}e^{2\pi i\xi z}e^{2\pi i\eta w}d\xi d\eta$$

If φ has only "low" frequencies and ψ has only "high frequencies", then (ξ, η) is far from the antidiagonal $\eta = -\xi$

$\implies$ $e^{-2\pi i(\xi+\eta)x}$ and $e^{-2\pi i(\xi+\eta)y}$ bring useful cancellation

Approach $\eta = -\xi$ using an appropriate L-P decomposition

Continuous scales are easier [Durcik (2014)]

Take a Gaussian cutoff function: $g(s) := e^{-\pi s^2}$

$$\Theta(F) := \sum_{j=1}^{m} \int_{\mathbb{R}^4} F(y,z)F(x,z)F(y,w)F(x,w)$$
$$g_{2^{k_j}}(z-w)(g_{2^{k_{j-1}}} - g_{2^{k_j}})(x-y)\,dxdydzdw$$

$$\widetilde{\Theta}(F) := \sum_{j=1}^{m} \int_{\mathbb{R}^4} F(y,z)F(x,z)F(y,w)F(x,w)$$
$$(g_{2^{k_j-1}} - g_{2^{k_j}})(z-w)g_{2^{k_{j-1}}}(x-y)\,dxdydzdw$$

Integration by parts and/or summation by parts:

$$\underbrace{\Theta(F)}_{\geq 0} + \underbrace{\widetilde{\Theta}(F)}_{\geq 0} = \underbrace{\text{a single scale quantity}}_{\text{easy to bound}}$$

Can we dominate a Schwartz function by the Gaussian function? Tail decaying too rapidly!

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We can dominate by a superposition of the Gaussians [Durcik (2014)]:

$$g(s) := e^{-\pi s^2}, \quad \sigma(s) := \int_1^\infty g_\alpha(s)\alpha^{-\lambda} d\alpha, \quad \lambda > 1$$

$$\lim_{|s|\to\infty} |s|^\lambda \sigma(s) \in (0,\infty) \implies (1+|s|)^{-\lambda} \lesssim_\lambda \sigma(s)$$

Proposition

Let $\lambda > 1$ and let $\vartheta, \varphi \in \mathcal{S}(\mathbb{R})$ be such that

$$|\vartheta(s)| \leq (1+|s|)^{-\lambda}, \quad |\varphi(s)| \leq (1+|s|)^{-\lambda}.$$

Assume that $\widehat{\vartheta}$ is supported in $[-2^{-4}, 2^{-4}]$, while $\widehat{\varphi}$ is supported in $[-1,1]$ and constant on $[-2^{-2}, 2^{-2}]$.
Then for any $m \in \mathbb{N}$, $k_0, \ldots, k_m \in \mathbb{Z}$, and any $F, G \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\Big| \sum_{j=1}^{m} \int_{\mathbb{R}^4} F(x+u,y)G(x,y+u)F(x+v,y)G(x,y+v) \vartheta_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)\, dxdydudv \Big| \lesssim_\lambda \|F\|_{L^4(\mathbb{R}^2)}^2 \|G\|_{L^4(\mathbb{R}^2)}^2.$$

Proposition

Let $\lambda > 1$ and let $\Phi \in \mathcal{S}(\mathbb{R}^2)$ be such that

$$|\Phi(u,v)| \leq (1+|u+v|)^{-\lambda}(1+|u-v|)^{-2\lambda}.$$

Assume that $\widehat{\Phi}$ is supported in $([-2,-2^{-5}] \cup [2^{-5},2])^2$.
Then for any $F, G \in \mathcal{S}(\mathbb{R}^2)$ and any $N \in \mathbb{N}$ we have

$$\Big| \sum_{j=-N}^{N} \int_{\mathbb{R}^4} F(x+u,y)G(x,y+u)F(x+v,y)G(x,y+v) \\ \Phi_{2^j}(u,v)\, dxdydudv \Big| \lesssim_\lambda \|F\|_{\mathrm{L}^4(\mathbb{R}^2)}^2 \|G\|_{\mathrm{L}^4(\mathbb{R}^2)}^2 .$$

Treating the long variation:

$$V(F,G) := \sum_{j=1}^{m} \big\| A^{\varphi}_{2^{k_j}}(F,G) - A^{\varphi}_{2^{k_{j-1}}}(F,G) \big\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$$

$$\varphi = \varphi * \chi + \varphi * \omega$$

$$\operatorname{supp}\widehat{\chi} \subseteq [-2^{-4}, 2^{-4}], \quad \operatorname{supp}\widehat{\omega} \subseteq [-2, -2^{-5}] \cup [2^{-5}, 2]$$

$$\Lambda(F,G) := \sum_{j=1}^{m} \int_{\mathbb{R}^4} F(x+u,y)G(x,y+u)F(x+v,y)G(x,y+v) \\ (\varphi * \chi)_{2^{k_j}}(u)(\varphi_{2^{k_j}} - \varphi_{2^{k_{j-1}}})(v)\, dxdydudv$$

$$\widetilde{\Lambda}(F,G) := \sum_{j=-N}^{N} \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}} F(x+u,y)G(x,y+u)(\varphi * \omega)_{2^j}(u)\, du \Big)^2 dxdy$$

A bootstrapping estimate:

$$V(F,G) \lesssim \|F\|^2_{\mathrm{L}^4(\mathbb{R}^2)} \|G\|^2_{\mathrm{L}^4(\mathbb{R}^2)} + |\Lambda(F,G)| + \widetilde{\Lambda}(F,G)^{1/2} V(F,G)^{1/2}$$

Thank you for your attention!