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The Harnack inequality in generalized Orlicz spaces

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Non-standard growth

Consider the minimization problem $\inf_u \int_\Omega F(x, |\nabla u|)\, dx$.

Standard growth

$F(x,t) \approx t^p$, $p \in (1,\infty)$

$F(x,t) \approx t^p w(x)$, $w \in A_p$

Non-standard growth

Marcellini *et al.* $t^p - 1 \lesssim F(x,t) \lesssim t^q + 1$, $p < q$

Variable exponent $F(x,t) \approx t^{p(x)}$

Double phase $\int |f|^p + a(x)|f|^q\, dx$

Generalized Orlicz $F(x,t) \approx \varphi(x,t)$

Generalized Orlicz spaces

Lebesgue -> Orlicz -> generalized Orlicz

$$\int |f|^p \, dx \quad \text{to} \quad \int \varphi(|f|) \, dx \quad \text{to} \quad \int \varphi(x, |f|) \, dx.$$

Or Lebesgue -> variable exponent -> generalized Orlicz

Studied since the 1940's; monograph by Musielak (1983)

Covers both variable exponent and Orlicz

Examples

Double phase minimization $\int_\Omega |f|^p + a(x)|f|^q\,dx$ studied in

- Baroni, Colombo, Mingione (NA 2015; SPMJ 2016)
- Colombo, Mingione (ARMA 2015; ARMA 2015; JFA 2016)

log-type $\int_\Omega |f|^{p(x)} \log(e + |f|)^{q(x)}\,dx$

- Spaces: Maeda, Mizuta, Nakai, Ohno, Shimomura (2008; 2010; 2010; 2011; 2012; 2015)
- Minimization: Giannetti and Passarelli di Napoli (JDE 2013); Jihoon Ok (CVPDE 2015)

Motivation

- Fluid dynamics models (Wróblewska-Kamińska, 2014)
- Existence of solutions to parabolic equations with generalized Orlicz growth (Świerczewska-Gwiazda, 2014)
- Renormalized solutions in generalized Orlicz spaces (Gwiazda, Wittbold, Wróblewskaa and Zimmermann, 2012, 2015)
- Regularity of minimizers (previous slide)
  - In the variable exponent case, the change in the anisotropy (growth rate) is gradual owing to the continuity of p. (Electrorheological fluids)
  - In other situations, such as composite materials, a more clear-cut transition is better. (Double-phase)

Harnack's inequality

Theorem (The Harnack inequality)

Assume that (A0), (A1), (A1-n), (aDec) and (aInc) hold. Suppose that $u \in W^{1,\varphi(\cdot)}_{\mathrm{loc}}(\Omega)$ is a local quasiminimizer. If $R < R_0$, then

$$\operatorname*{ess\,sup}_{Q_R} u \leqslant C \operatorname*{ess\,inf}_{Q_R} u + R$$

Corollary

Assume that (A0), (A1), (A1-n), (aDec) and (aInc) hold. If u is a local quasiminimizer, then $u \in C^{\alpha}_{\mathrm{loc}}(\Omega)$.

Joint work with Petteri Harjulehto (Turku) and Olli Toivanen (Warsaw)

Local quasiminimizers

We say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local K-quasiminimizer if

$$\int_\Omega F(x, |\nabla u|)\, dx \leqslant K \int_\Omega F(x, |\nabla v|)\, dx$$

for all $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with $\operatorname{spt}(u - v) \Subset \Omega$.

Assumptions

$\varphi_B^+(t) := \sup_{x\in B} \varphi(x,t)$ and $\varphi_B^-(t) := \inf_{x\in B} \varphi(x,t)$

(A0) There exists $\beta \in (0,1)$ such that $\varphi(x,\beta) \leqslant 1 \leqslant \varphi(x,1)$ for every $x \in \Omega$.

(A1) There exists $\beta \in (0,1)$ such that, for every ball $B \subset \Omega$,

$$\varphi_B^+(\beta t) \leqslant \varphi_B^-(t) \quad \text{when} \quad t \in \Big[1, (\varphi_B^-)^{-1}(\tfrac{1}{|B|})\Big].$$

(A1-n) There exists $\beta \in (0,1)$ such that, for every ball $B \subset \Omega$,

$$\varphi_B^+(\beta t) \leqslant \varphi_B^-(t) \quad \text{when} \quad t \in \Big[1, \tfrac{1}{|B|^{1/n}}\Big].$$

(aInc) There exists $\gamma^- > 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^{\gamma^-}}$ is almost increasing for $t > 0$ uniformly in Ω.

(aDec) There exists $\gamma^+ > 1$ such that $t \mapsto \frac{\varphi(x,t)}{t^{\gamma^+}}$ is almost decreasing for $t > 0$ uniformly in Ω.

Assumptions, variable exponent case

$\varphi(x,t) := t^{p(x)} w(x)$

(A0) $w \approx 1$.

(A1) p is locally log-Hölder continuous

(A1-n) p is locally log-Hölder continuous

(aInc) $p^- > 1$

(aDec) $p^+ < \infty$

Assumptions, double phase case

$\varphi(x,t) := t^p + a(x)t^q$, $q > p$

(A0) $a \in L^\infty$.

(A1) $a \in C^\alpha$, $\frac{q}{p} \leqslant 1 + \frac{\alpha}{n}$

(A1-n) $a \in C^\alpha$, $q \leqslant p + \alpha$

(aInc) $p > 1$

(aDec) $q < \infty$

If $\varphi(x,t) := t^p + a(x)t^p \log(e+t)$, then (A1) follows from log-Hölder continuity.

Other results

(A0)–(A2) + (aInc) $\Rightarrow$ maximal operator bounded (JFA)

(A0)–(A2) + (aInc) + $\gamma^+ < n$ $\Rightarrow$ Riesz potential operator bounded (Harjulehto, FM)

(A0)–(A2) + (aDec) $\Rightarrow$ extrapolation (Cruz-Uribe)

Ideas for proof of Harnack inequality

Caccioppoli estimate straight-forward

$$\int_{A(k,r)} \varphi(x, |\nabla(u-k)_+|)\,dx \lesssim \int_{A(k,R)} \varphi\left(x, \frac{u-k}{R-r}\right)\,dx,$$

The difficulty is mixing modular and norm. Auxiliary estimates:

$$\int_Q \varphi(v)^{n'} \lesssim \left(\int_Q \varphi(\nabla v)\right)^{n'} \quad ; \quad \int_Q \varphi(x, Mf_1)\,dx \lesssim \int_Q \varphi(x, f_1).$$

Split small and large f;
$\bar{\varphi}(x,t) := \varphi_Q^-(t)\chi_{[0,1]}(t) + \varphi(x,t)\chi_{(1,\infty)}(t)$

$$\int_{Q_\sigma} \bar{\varphi}(x,(u-k)_+) \lesssim \int_{Q_\tau} \bar{\varphi}\left(x, \frac{(u-h)_+}{\tau-\sigma}\right)\left(\int_{Q_\tau} \frac{\bar{\varphi}(x,(u-h)_+)}{\bar{\varphi}(x,k-h)}\right)^{\alpha}.$$

Ideas for proof, continued

$$\int_{Q_\sigma} \bar\varphi(x,(u-k)_+) \lesssim \int_{Q_\tau} \bar\varphi\left(x, \frac{(u-h)_+}{\tau-\sigma}\right) \left(\int_{Q_\tau} \frac{\bar\varphi(x,(u-h)_+)}{\bar\varphi(x,k-h)} \right)^{\alpha}$$

by the usual iteration, inhomogeneous version:

$$\operatorname*{ess\,sup}_{Q_{R/2}} u-k \lesssim R^{-\frac{\gamma^+}{\alpha\gamma^-}} \left(\int_{Q_R} \varphi(x,(u-k)_+) \right)^{\frac{1}{\gamma^-}} + 1$$

Minimizer is locally bounded, scaling $u_\delta(x) := \frac{1}{\delta} u(\delta x)$ and (A1-n) give

$$\operatorname*{ess\,sup}_{Q_{R/2}} u-k \lesssim \left(\fint_{Q_R} (u-k)_+^{\gamma^+} \right)^{\frac{1}{\gamma^+}} + R$$

The inf-estimate is fairly standard, with Krylov–Safonov lemma.

Boundedness of maximal operator

The boundedness of the maximal operator in generalized Orlicz spaces was shown by Maeda, Mizuta, Ohno and Shimomura (2013) in $L^{\varphi(\cdot)}(\mathbb{R}^n)$ assuming (A0), (A1-1), (A2), (aInc) and (aDec).

Improved by me (in JFA, 2015; 2016) to (A0), (A1), (A2) and (aInc), and with Petteri Harjulehto (2016) to $L^{\varphi(\cdot)}(\Omega)$ (condition for extension).

Proof of maximal operator result.

Let $\psi \in \Phi$ be as in the lemma. It suffices to show that $M : L^\psi(\mathbb{R}^n) \to L^\psi(\mathbb{R}^n)$. Since $\psi^{1/\gamma}$ is convex, it follows from Jensen's inequality that

$$\psi(\epsilon Mf) = \big(\psi^{\frac{1}{\gamma}}(\epsilon Mf)\big)^\gamma \leqslant \big(M\big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)\big)^\gamma.$$

Let $f \in L^\psi(\mathbb{R}^n)$ and $\epsilon := \|f\|_\psi^{-1}$ so that $\varrho_\psi(\epsilon f) \leqslant 1$. Since M is bounded in $L^\gamma(\mathbb{R}^n)$, we obtain that

$$\int_{\mathbb{R}^n} \big(M\big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)\big)^\gamma \, dx \lesssim \int_{\mathbb{R}^n} \big(\psi^{\frac{1}{\gamma}}(\epsilon f)\big)^\gamma \, dx = \int_{\mathbb{R}^n} \psi(\epsilon f) \, dx \leqslant 1.$$

Hence $\varrho_\psi(\epsilon Mf) \lesssim 1$, which implies that $\|\epsilon Mf\|_\psi \lesssim 1$. Dividing by ϵ, we find that $\|Mf\|_\psi \lesssim \frac{1}{\epsilon} = \|f\|_\psi$, which completes the proof. □

Auxiliary results

Shimomura et al.'s results are included, on account of the following lemma.

Lemma

Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is doubling and that $s \mapsto \frac{\varphi(s)}{s}$ is increasing. Then φ is equivalent to a convex function $\psi \in \Phi$.

Every Φ-function satisfying (A0)–(A2) is equivalent to a normalized Φ-function.

Definition

We say that $\varphi \in \Phi_1(\mathbb{R}^n)$ is a *normalized Φ-function* if $\varphi(x, t) = \varphi_\infty(t)$ for $t \in [0, 1]$ and there exists $\beta > 0$ such that

$$\beta\varphi^{-1}(x, t) \leqslant \varphi^{-1}(y, t)$$

for every $t \in \left[0, \frac{1}{|B|}\right]$, every $x, y \in B$ and every ball B.

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