Parabolic BMO and the forward-in-time maximal operator

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Singular integrals and partial differential equations
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BMO

- A function $u \in L^1_{loc}$ is said to be of *bounded mean oscillation* ($u \in \mathrm{BMO}$) if

$$\|u\|_{\mathrm{BMO}} = \sup_Q \fint_Q |u - u_Q| < \infty.$$

- The remarkable John-Nirenberg inequality asserts that

$$\sup_Q \fint_Q \exp(\epsilon |u - u_Q|) < \infty$$

for some positive $\epsilon \lesssim \|u\|_{\mathrm{BMO}}^{-1}$.

- BMO is connected to many questions of harmonic analysis.
- There is also an interesting connection between BMO and the regularity theory of elliptic PDE of divergence form.

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Elliptic PDE

- Let A be a matrix of measurable functions $a_{ij}(x)$ such that

$$\Lambda^{-1}|\xi|^2 \leq \xi \cdot A\xi \leq \Lambda|\xi|^2$$

for some $\Lambda \in (1, \infty)$ uniformly in x.

- If w is a positive weak (super)solution to

$$\operatorname{div}(A\nabla w) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n,$$

then $u = \log w \in \mathrm{BMO}(\Omega)$. This is an important observation in Moser's proof of the DeGiorgi–Nash–Moser theorem.

- As a consequence, $w^\epsilon \in A_2$. It is also true that $w \in A_1$.
- Recall that $w \in A_p$ if

$$[w]_{A^p} = \sup_{Q \subset \Omega} \fint_Q w \left(\fint_Q w^{1-p'} \right)^{p-1} < \infty, \quad 1 \leq p \leq \infty.$$

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Parabolic PDE

- We would like to study BMO in the context of parabolic differential equations.
- We consider local solutions to e.g. one of the following

$$u_t - \Delta u = 0,$$
$$u_t - \operatorname{div}(A\nabla u) = 0,$$
$$(u^{p-1})_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

  in $\Omega \times (0, T)$. For our purposes, the last one is the most general one, and we will concentrate on it.
- In general, the positive solutions cannot be Muckenhoupt A_2 weights in any obvious way (they can fail to be doubling measures with respect to any reasonable metric). Consequently, the parabolic BMO must be something non-trivial.

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Plan of the talk

I will give a summary of the recent results about parabolic BMO. Some of them are joint with J. Kinnunen. I will discuss

1. Notation in the space time $\mathbb{R}^{n+1}$.
2. The definition of parabolic BMO.
3. Weights.
4. The forward-in-time maximal operator.

Notation

- The basic structure of $u_t - \Delta u = 0$ and its generalizations is preserved under translations $z \mapsto z + h$ and anisotropic dilations $(x, t) \mapsto (\delta x, \delta^p t)$ of the coordinates. ($p = 2$ for the heat equation)
- These transformations generate parabolic rectangles. We denote

$$R = R(x,t,L) = Q(x,L) \times (t - L^p, t + L^p),$$
$$R^+(\gamma) = Q(x,L) \times (t + \gamma L^p, t + L^p) \quad \text{and}$$
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The starting point in PDE

- It was discovered in the 1960s that the solutions to parabolic equations f satisfy

$$\fint_{R^+(0)} \fint_{R^-(0)} \sqrt{(u(x) - u(y))^+} \, \mathrm{d}x \, \mathrm{d}y < C(n, p)$$

for $u = -\log f$. (Moser, Trudinger)

- The parabolic John-Nirenberg lemma (Moser, Trudinger, Aimar) tells that

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The definition of PBMO$^-$, S. 2014

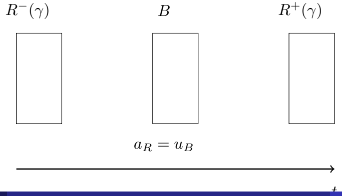
- We define that $u \in \text{PBMO}^-$ if

$$\|u\|_{\text{PBMO}^-} := \sup_R \inf_a \left(\fint_{R^-(\frac{1}{2})} (u-a)^+ + \fint_{R^+(\frac{1}{2})} (a-u)^+ \right) < \infty.$$

- It holds that

$$\|u\|_{\text{PBMO}^-} \eqsim_{n,p,\gamma} \sup_R \inf_a \left(\fint_{R^-(\gamma)} (u-a)^+ + \fint_{R^+(\gamma)} (a-u)^+ \right).$$

- It is possible to replace the constant a by a mean value in a certain cylinder:



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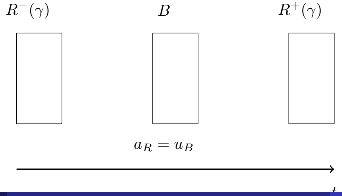

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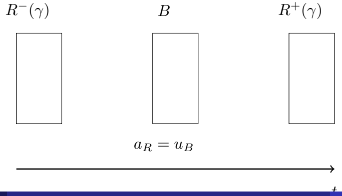

Weights I

- The weights $A_q^+(\gamma)$ corresponding to PBMO^- are the ones satisfying

$$\sup_R \fint_{R^-(\gamma)} w \left(\fint_{R^+(\gamma)} w^{1-q'} \right)^{q-1} < \infty, \quad 1 < q < \infty.$$

- As in the case of PBMO^-, we have that $A_q^+(\gamma) = A_q^+(\gamma')$ for all $\gamma, \gamma' \in (0,1)$.
- It holds

$$\mathrm{PBMO}^- = \{\alpha \log w : w \in A_q^+, \alpha \in (0,\infty), q \in (1,\infty)\}.$$

(Kinnunen and S. 2014)

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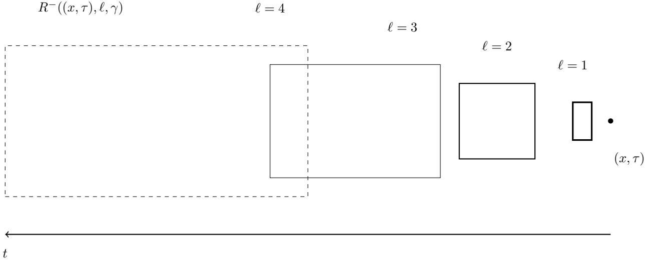
Weights II

- We define the forward-in-time maximal function as

$$M^{\gamma +} f(z) := \sup_{\ell > 0} \fint_{R^+(z,\ell,\gamma)} |f|.$$

- For $q \in (1, \infty)$, the operator $M^{\gamma +} : L^q(w) \to L^q(w)$ is bounded if and only if $w \in A_q^+(\gamma)$ (Kinnunen and S. 2014).



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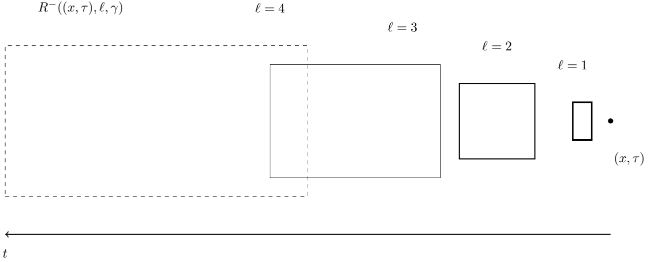

Theorem

Let $u \in \mathrm{PBMO}^+$ be non-negative. If $M^{\gamma+}u \in L^1_{loc}$, then $M^{\gamma+}u \in \mathrm{PBMO}^+$.

- The theorem holds true in $\mathbb{R}^{n+1}$ and $\Omega \times \mathbb{R}$.
- Nonnegativity is necessary at least in the latter case.

PBMO and forward-in-time maximal function, S. 2016

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Remarks

- It may be the that the maximal function

$$M_*^{\gamma +} u(z) = \sup_{R(z)} ((u^+)_{R^+(\gamma)} + (u^-)_{R^-(\gamma)})$$

  is more correct object than what was studied previously. They coincide for positive functions.
- $M_*^{\gamma +}$ almost maps $\text{PBMO}^+ \to \text{PBMO}^+ + \text{PBMO}^-$ (compare to the classical BMO $\to$ BMO).
- There exists a class of functions playing the role of "parabolic Hardy space H^1" in the sense of "duality".

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