Weighted norm inequalities of $(\mathbf{1}, \mathbf{q})$-type for integral operators and a sublinear version of Schur's lemma

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Abstract

We will present characterizations of strong-type and weak-type weighted norm inequalities of $(1, q)$-type, for integral operators with positive kernels (in particular, Greens kernels) and maximal functions.

These inequalities are motivated by applications to certain nonlinear elliptic PDE and convolution equations with radially symmetric non-decreasing kernels.

Our approach relies on a **sublinear analogue of Schur's lemma** for integral operators with positive kernels which satisfy the **weak maximum principle**.

In particular, we obtain existence and global pointwise estimates of solutions to "sublinear" elliptic problems of the type

$$-\triangle u - \sigma\, u^q = 0 \text{ in } \Omega \subseteq R^n, \quad 0 < q \leq 1,$$

where $0 \leq \sigma \in L^1_{loc}(\Omega)$, or a positive measure $\sigma \in \mathcal{M}^+(\Omega)$.
Based on joint work with Dat Tien Cao and with Stephen Quinn.

Extensions: quasilinear, fully nonlinear, fractional Laplacian

- Quasilinear equations and inequalities:
  We will consider inequalities of the type:

$$-\triangle_p u - \sigma\, u^q \geq 0,$$

  $\triangle_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $0 < q < p-1$ (**sub-natural growth**). The case $q = p-1$ corresponds to the **natural growth** case [Jaye-Verbitsky, '12]; $q > p-1$ **"super-natural" growth** [Phuc-Verbitsky, '08, '09].
- Fully nonlinear k-Hessian equations and inequalities:

$$F_k[u] - \sigma\, |u|^q \geq 0,$$

  in the class of k-convex functions u, $0 < q < k$ (**sub-natural growth**). $F_k[u] =$ the sum of the $k \times k$ principal minors of the Hessian matrix $D^2 u$.
- Fractional Laplacian $(-\triangle)^{\frac{\alpha}{2}}$, subelliptic Laplacian, etc.

Publications

1. *Global estimates for kernels of Neumann series and Greens functions* (with Michael Frazier and Fedor Nazarov) **J. London Math. Soc., 90 (2014) 903–918**
2. *Positive solutions to Schrödinger's equation and the exponential integrability of the balayage* (with Michael Frazier) **arXiv:1509.09005**
3. *Finite energy solutions of quasilinear elliptic equations with sub-natural growth terms* (with Dat Tien Cao) **Calc. Var. PDE, 52 (2015), 529–546**
4. *Nonlinear elliptic equations and intrinsic potentials of Wolff type* (with Dat Tien Cao) **arXiv:1409.4076**
5. *Pointwise estimates of Brezis–Kamin type for solutions of sublinear elliptic equations* (with Dat Tien Cao) **arXiv:1601.05496**
6. *Weighted norm inequalities of* $(\mathbf{1}, \mathbf{q})$*-type for integral and fractional maximal operators* (with Stephen Quinn) **arXiv:1606.03794**

Motivation: linear and nonlinear elliptic PDE

Classical time-independent Schrödinger equations, sublinear elliptic problems

Let $\Omega \subseteq \mathbf{R}^n$ be a domain with a positive Green's function $\mathbf{G}^{\Omega}(\mathbf{x}, \mathbf{y})$.

Linear elliptic problems: Find conditions for the existence, and global pointwise estimates of positive solutions $\mathbf{u}$ to the Dirichlet problem:

$$-\Delta \mathbf{u} = \sigma\, \mathbf{u} + \varphi \text{ on } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega,$$

for general nonnegative σ, φ, $\mathbf{g}$ (measurable functions, possibly measures).

Sublinear elliptic problems: For $0 < \mathbf{q} < 1$, study $\mathbf{u} > 0$ such that

$$-\Delta \mathbf{u} = \sigma\, \mathbf{u}^{\mathbf{q}} + \varphi \text{ on } \Omega, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\Omega,$$

for general nonnegative σ, φ, $\mathbf{g}$.

Main idea: Use the underlying weighted norm inequalities for Green's operators

$$\mathbf{G}\colon \mathbf{L}^2(\Omega, \sigma) \to \mathbf{L}^2(\Omega, \sigma) \quad \text{in the linear case;}$$

$$\mathbf{G}\colon \mathbf{L}^1(\Omega, d\mathbf{x}) \to \mathbf{L}^{\mathbf{q}}(\Omega, \sigma) \quad \text{in the sublinear case.}$$

Part I: Linear integral equations

Consider the integral operator

$$\mathbf{Tf(x) = \int_{\Omega} K(x,y)\, f(y)\, d\sigma(y)}$$

on a $\boldsymbol{\sigma}$-finite measure space $\mathbf{(\Omega, \sigma)}$; $\mathbf{K(x,y) \geq 0}$, $\mathbf{f \geq 0}$. Consider the formal Neumann series

$$\mathbf{(I - T)^{-1} = I + \sum_{j=1}^{\infty} T^j}$$

and the kernel of the resolvent $\mathbf{(I - T)^{-1}}$. Define the associated kernels $\mathbf{K_1 = K}$ and

$$\mathbf{K_j(x,y) = \int_{\Omega} K_{j-1}(x,z)\, K(z,y)\, d\sigma(z)}$$

for $\mathbf{j \geq 2}$, of the operators $\mathbf{T^j}$. Define the formal Green's function

$$\mathbf{V(x,y) = \sum_{j=1}^{\infty} K_j(x,y).}$$

Quasi-metric kernels

Let (Ω, σ) be a measure space. Suppose $K : \Omega \times \Omega \longrightarrow (0, +\infty]$, where $K(\cdot, y)$ is σ-measurable for all $y \in \Omega$. Define $d(x, y) = 1/K(x, y)$. We say that K is a *quasi-metric* kernel on Ω (with quasi-metric constant $\kappa > 0$) if:

(i) K is symmetric: $K(x, y) = K(y, x)$ for all $x, y \in \Omega$;
(ii) d satisfies the quasi-triangle inequality with constant κ:

$$d(x, y) \le \kappa \, (d(x, z) + d(z, y))$$

for all $x, y, z \in \Omega$.

Remarks: 1. Do not need $d(\cdot, \cdot) = 0$ on the diagonal.
2. Enough to assume in (i) that K is merely quasi-symmetric.
3. $d(x, y) \approx \rho(x, y)^{\beta}$, $\beta = \beta(\kappa)$; ρ satisfies the triangle inequality (Aoki-Rolewicz for **linear** quasi-metric spaces; Heinonen, larger constant).

Key Abstract Theorem

Let $\|\mathbf{T}\| = \|\mathbf{T}\|_{L^2(\sigma)\to L^2(\sigma)}$ denote the operator norm of $\mathbf{T}$.
If $\|\mathbf{T}\| > 1$, then one has $\mathbf{V}(\cdot, \mathbf{y}) = +\infty$ a.e.

Theorem (Frazier-Nazarov-Verbitsky, '14)

Let (Ω, σ) be a σ-finite measure space. Let $\mathbf{K}$ be a quasi-metric kernel. Suppose $\|\mathbf{T}\| < 1$. Then there exists $\mathbf{c} > 0$ depending only on κ, and $\mathbf{C} > 0$ depending only on κ and $\|\mathbf{T}\|$ such that

$$K(x, y)e^{cK_2(x,y)/K(x,y)} \le V(x, y) \le K(x, y)e^{CK_2(x,y)/K(x,y)}.$$

In the critical case $\|\mathbf{T}\| = 1$, the lower bound still holds, but there are examples where the upper bound holds and also examples where $\mathbf{V}(\cdot, \mathbf{y}) = +\infty$ a.e. for every $\mathbf{y}$, but $\mathbf{K_2}(\cdot, \mathbf{y}) < +\infty$ a.e.

Weak Boundedness Condition

Quasi-metric balls: $\mathbf{B(x, r) = \{y \in \Omega : d(x, y) < r\}}$.

Theorem (Nazarov)

Let $(\boldsymbol{\Omega}, \boldsymbol{\sigma})$*, and* $\mathbf{T}$ *be as above. Then*

$$\mathbf{T : L^2(\Omega, d\sigma) \to L^2(\Omega, d\sigma)}$$

is a bounded operator if and only if there exists a constant $\mathbf{c = c(\kappa) > 0}$ *such that, for every* $\mathbf{x \in \Omega,\ r > 0}$*,*

(A) $\boldsymbol{\sigma}\mathbf{(B(x, r)) \le c\, r}$*,*

(B) $\displaystyle \iint_{\mathbf{B(x,r) \times B(x,r)}} \mathbf{K(y, z)\, d\sigma(y)\, d\sigma(z) \le c\, \sigma(B(x, r))}$*.*

Moreover, $||\mathbf{T}||$ *is equivalent to the least constant* $\mathbf{c}$ *such that both (A) and (B) hold.*

Proof: some ideas of Guy David, non-homogeneous harmonic analysis.

Part II: Sublinear integral equations

Key weighted norm inequalities of $(1, q)$-type in the case $0 < q < 1$:

$$\|G\nu\|_{L^q(\Omega, d\sigma)} \le C\,\|\nu\|, \tag{1}$$

for all locally finite positive measures $\nu \in \mathcal{M}^+(\Omega)$, $\|\nu\|_{\mathcal{M}^+(\Omega)} = \nu(\Omega)$, and G is the integral operator with nonnegative kernel,

$$G\nu(x) = \int_\Omega G(x, y)\, d\nu(y).$$

Weak-type weighted norm inequalities of $(1, q)$-type, $0 < q \le 1$:

$$\|G\nu\|_{L^{q,\infty}(\Omega, d\sigma)} \le C\,\|\nu\|, \tag{2}$$

for all positive measures $\nu \in \mathcal{M}^+(\Omega)$.

Remarks: 1. One can use $L^1(\Omega)$ in place of $\mathcal{M}^+(\Omega)$ in both (1) and (2).
2. The case $q = 1$ in (2) is related to the previous theorem.

Sublinear integral equations

The study of $(1, q)$ weighted norm inequalities for for $0 < q < 1$ is motivated by applications to sublinear elliptic PDE of the type

$$\begin{cases} -\Delta u = \sigma\, u^q & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \iff \quad u = G(u^q d\sigma),$$

where $u > 0$, for $0 < q < 1$, $\sigma \in \mathcal{M}^+(\Omega)$; $\Omega \subseteq \mathbf{R}^n$ open set with non-trivial Green's function $G(x, y)$.

The only restrictions imposed on the kernel G:
(a) G is quasi-symmetric, $a^{-1}\, G(x, y) \le G(y, x) \le a\, G(x, y)$;
(b) G satisfies a weak maximum principle (WMP).

In particular, G can be a Green operator associated with the Laplacian, or a more general elliptic operator (including the fractional Laplacian), or a convolution operator on $\mathbf{R}^n$ with radially symmetric decreasing kernel $G(x, y) = k(|x - y|)$.

Conditions on kernels of integral operators

Let $\mathbf{G}\colon \Omega \times \Omega \to [0, +\infty]$ be a nonnegative lower-semicontinuous kernel.

Definition

$\mathbf{G}$ is a quasi-symmetric kernel if there exists a constant $a > 0$ such that

$$a^{-1}\, G(x,y) \le G(y,x) \le a\, G(x,y), \quad x, y \in \Omega.$$

Definition

$\mathbf{G}$ is *degenerate* with respect to $\sigma \in \mathcal{M}^+(\Omega)$ if there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ such that

$$G(\cdot, y) = 0 \quad d\sigma\text{-a.e. for } y \in A.$$

Otherwise, $\mathbf{G}$ is *non-degenerate* with respect to σ.

See [Sinnamon, '05] in the context of Schur's lemma for positive operators $T\colon L^p \to L^q$ in the case $1 < q < p$.)

The weak and strong maximum principles

If $\nu \in \mathcal{M}^+(\Omega)$, then by $\mathbf{G}\nu$ and $\mathbf{G}^*\nu$ we denote the potentials

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y)\, d\nu(y), \quad \mathbf{G}^*\nu(x) = \int_\Omega G(y, x)\, d\nu(y), \quad x \in \Omega.$$

Definition

$\mathbf{G}$ satisfies the *weak maximum principle* (WMP) if, for any constant $M > 0$, and $\nu \in \mathcal{M}^+(\Omega)$, the inequality

$$\mathbf{G}\nu(x) \le M \quad \text{for all} \;\; x \in S(\nu)$$

implies

$$\mathbf{G}\nu(x) \le h\, M \quad \text{for all} \;\; x \in \Omega,$$

where $h \ge 1$ is a constant, and $S(\nu) = \operatorname{supp} \nu$.
If $h = 1$, then $\mathbf{G}$ satisfies the *strong maximum principle* (MP).

Green's kernels are often (quasi-)symmetric, and (MP) holds [Ancona, '02].

Potential Theory

Capacities and contents

Let $\mathbf{K} \subset \mathbf{X}$ be a compact set. For the kernel $\mathbf{G} : \mathbf{X} \times \mathbf{Y} \to [0, +\infty]$, consider several different related notions of **capacity/content:**

$$\mathbf{cap_0(K) = sup} \left\{ \mu(\mathbf{K}): \ \mu \in \mathcal{M}^+(\mathbf{K}), \quad \mathbf{G}^*\mu(\mathbf{y}) \le \mathbf{1}, \ \forall \ \mathbf{y} \in \mathbf{Y} \right\},$$

$$\mathbf{cont(K) = inf} \left\{ \lambda(\mathbf{Y}): \ \lambda \in \mathcal{M}^+(\mathbf{Y}), \quad \mathbf{G}\lambda(\mathbf{x}) \ge \mathbf{1}, \ \forall \ \mathbf{x} \in \mathbf{K} \right\}.$$

These two notions in fact coincide [Fuglede '65] via von Neumann's minimax theorem. For $\mathbf{X} = \mathbf{Y} = \mathbf{\Omega}$, the **Wiener capacity** is defined by

$$\mathbf{cap(K) = sup} \left\{ \mu(\mathbf{K}): \mu \in \mathcal{M}^+(\mathbf{K}), \quad \mathbf{G}^*\mu(\mathbf{y}) \le \mathbf{1}, \ \forall \ \mathbf{y} \in \mathbf{S}_\mu \right\}.$$

Note that $\mathbf{cap_0(K)} \le \mathbf{cap(K)} \le \mathbf{h\, cap_0(K)}$, if $\mathbf{G}$ satisfies (WMP) for the upper estimate. The Wiener capacity is most useful (quasi-symmetric $\mathbf{G}$).

Weak-type $(1, q)$-inequality for integral operators

Theorem

Let $\sigma \in \mathcal{M}^+(\Omega)$, and $0 < q \leq 1$. Then the following statements are equivalent:

1. *There exists a constant $\varkappa_w > 0$ such that*

$$||G\nu||_{L^{q,\infty}(\sigma)} \leq \varkappa_w ||\nu||, \quad \forall\, \nu \in \mathcal{M}^+(\Omega).$$

2. *There exists a constant $c > 0$ such that*

$$\sigma(K) \leq c\Big(\mathrm{cap}_0(K)\Big)^q, \quad \forall \text{ compact sets } K \subset \Omega.$$

3. *The condition $G\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$ holds, provided $0 < q < 1$, and G is quasi-symmetric and satisfies the (WMP).*

Remark. For $q > 1$: Obviously $(2) \Leftrightarrow \sup_{x\in\Omega} ||G(x,\cdot)||_{L^{q,\infty}(\sigma)} < \infty$ $\Leftrightarrow K = B(x,r)$ in (2) if G is quasi-metric $\Leftrightarrow \sigma(B(x,r)) \leq c\, r^q$.

Main Theorem in the sublinear case

Theorem

Let $\sigma \in \mathcal{M}^+(\Omega)$, and $0 < q < 1$. Suppose G is quasi-symmetric and satisfies the (WMP). Then the following statements are equivalent:

1. *There exists a constant $\varkappa(\sigma) > 0$ such that*

$$\|G\nu\|_{L^q(\sigma)} \leq \varkappa(\sigma)\, \|\nu\| \quad \forall \nu \in \mathcal{M}^+(\Omega).$$

2. *There exists a non-trivial supersolution $u \geq G(u^q d\sigma)$, $u \in L^q(\Omega, d\sigma)$.*
3. *There exists a positive solution $u = G(u^q d\sigma)$, $u \in L^q(\Omega, d\sigma)$, provided G is non-degenerate with respect to σ.*

Remarks. 1. The implication $(1)\Rightarrow(2)$ in the Theorem holds for any **G**.
2. The implications (2) or $(3)\Rightarrow(1)$ generally fail without the (WMP).
3. A **minimal** solution $u = \lim u_j$ is constructed **explicitly** by
$u_{j+1} = G(u_j^q d\sigma)$, $u_{j+1} \geq u_j$, $u_0 = c(G\sigma)^{\frac{1}{1-q}}$, c is a small constant.

Necessary/sufficient conditions for $(1, q)$-inequality

Lemma (Gagliardo '65)

Let $\sigma \in \mathcal{M}^+(\Omega)$, and let G be a positive kernel. Suppose the $(1, q)$-weighted norm inequality (1) holds, Then for every $\epsilon > 0$, there is a positive supersolution $u \in L^q(\sigma)$ such that

$$u \geq G(u^q \sigma)$$

with $\|u\|^q_{L^q(\sigma)} \leq (1+\epsilon)^{\frac{1}{1-q}} [\varkappa(\sigma)]^{\frac{q}{1-q}}$.

Remarks. **1.** In general, the Lemma fails if $\epsilon = 0$.
2. For non-degenerate G, in fact $\epsilon = 0$, and $u = G(u^q \sigma)$.
3. The converse fails without the (WMP) for any $1 + \epsilon > 0$.

Sufficient/necessary conditions for $(1, q)$-inequality

Lemma

Let $\sigma \in \mathcal{M}^+(\Omega)$, and let G be a quasi-symmetric kernel satisfying the (WMP). Let $0 < q < 1$. Then the $(1, q)$-weighted norm inequality (1) holds if $G\sigma \in L^{\frac{q}{1-q}, q}(\Omega, \sigma)$.

Remarks. 1. The exponents $\frac{q}{1-q}$ and q are sharp: the inequality fails if $G\sigma \in L^{s,r}(\Omega, \sigma)$ with $s = \frac{q}{1-q}$ and $r > q$, or $0 < s < \frac{q}{1-q}$, $r > 0$.

2. The condition $G\sigma \in L^{\frac{q}{1-q}}(\Omega, \sigma)$ is **necessary** for (1), but $G\sigma \in L^{s,r}(\Omega, \sigma)$ with $s = \frac{q}{1-q}$ and $r > \frac{q}{1-q}$ is not.

3. Another (independent) **necessary** condition is

$$\sup_{x \in \Omega} \int_\Omega G(x, y)^q d\sigma(y) < \infty.$$

Necessary conditions for $(1, q)$-inequality

We consider relations between the existence of positive (super)solutions $u \geq G(u^q d\sigma)$ and and the $(1, q)$-weighted norm inequality (1), as well as energy conditions of the type

$$\int_\Omega (G\sigma)^s d\sigma < \infty, \quad s > 0. \tag{3}$$

Lemma

Let $0 < q < 1$, and let $\sigma \in \mathcal{M}^+(\Omega)$. If inequality (1) holds, then (3) holds with $s = \frac{q}{1-q}$.

Corollary

If G is quasi-symmetric and the (WMP) holds, then $\int_\Omega (G\sigma)^{\frac{q}{1-q}} d\sigma < \infty$ is necessary for the existence of a positive supersolution $u \in L^q(\Omega, \sigma)$.

Necessary conditions for positive supersolutions

Let $\mathbf{q_0} = \frac{\sqrt{5}-1}{2} \approx 0.61\ldots$ denote the conjugate golden ratio.

Lemma

(a) Let $0 < q \le q_0$, *and let* $\sigma \in \mathcal{M}^+(\Omega)$. *If there exists a positive supersolution* $u \in L^q(\Omega, \sigma)$, *then* (3) *holds with* $s = \frac{q}{1-q}$.
(b) If $q_0 < q < 1$, *and* σ *is a* **finite** *measure, then* (3) *holds for* $0 < s \le 1 + q$.
(c) For $q_0 < q < 1$, *statement (a) generally fails, i.e., there exists a kernel* G *and measure* σ *such that there is a positive supersolution* $u \in L^q(\Omega, \sigma)$, *but* $\int_\Omega (G\sigma)^{\frac{q}{1-q}} d\sigma = +\infty$.
(d) The exponents $s = \frac{q}{1-q}$ *and* $s = 1 + q$ *in statements (a) and (b) respectively are sharp: there exist* G *for which* (3) *fails if* $s \ne \frac{q}{1-q}$ *in the case of general measures* σ, *and if* $s > \min\left(\frac{q}{1-q}, 1+q\right)$ *in the case of finite measures* σ.

Lower estimates for positive supersolutions

The necessity of the condition $\int_\Omega (G\sigma)^{\frac{q}{1-q}} d\sigma < \infty$ for the existence of a supersolution,

$$u(x) \ge G(u^q d\sigma)(x) \quad d\sigma\text{—a.e.}, \quad u \in L^q(\Omega, \sigma),$$

is immediate from the following theorem.

Theorem

Suppose G *is a symmetric kernel on* Ω *satisfying the weak maximum principle with constant* $h > 0$*. Let* $0 < q < 1$*. If* $u \ge 0$ *is a non-trivial supersolution, then*

$$u(x) \ge h^{-\frac{q}{(1-q)^2}} (1-q)^{\frac{1}{1-q}} \left[G\sigma(x) \right]^{\frac{1}{1-q}} \quad d\sigma\text{—a.e. in } \Omega.$$

Remark. The constant $(1-q)^{\frac{1}{1-q}}$ is sharp [Grigor'yan-V. '15].

Some counter examples

1. Suppose $\Omega = \mathbf{R^n}$, and $\mathbf{G(x, y) = |x - y|^{\alpha - n}}$ is the Riesz kernel of order $\alpha \in (\mathbf{0}, \mathbf{n})$. Then there exists a measure $\boldsymbol{\sigma}$ such that both necessary conditions mentioned above:

$$\int_\Omega (G\sigma)^{\frac{q}{1-q}} d\sigma < \infty, \quad \sup_{x \in \Omega} \int_\Omega G(x, y)^q d\sigma(y) < \infty$$

hold, but the $(\mathbf{1}, \mathbf{q})$-weighted norm inequality (1) fails.

2. There exists a positive non-degenerate symmetric kernel (matrix) **G** and measure $\boldsymbol{\sigma}$ on a discrete set $\boldsymbol{\Omega}$ such that there exists a positive supersolution $\mathbf{u} \in \mathbf{L^q}(\boldsymbol{\Omega}, \boldsymbol{\sigma})$, but the $(\mathbf{1}, \mathbf{q})$-weighted norm inequality (1) fails due to the lack of the (WMP).

Fractional Maximal Operators

Let $\nu \in \mathcal{M}^+(\mathbf{R}^n)$, and let $0 \le \alpha < n$. Define the fractional maximal operator M_α by

$$M_\alpha \nu(x) = \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|^{1-\frac{\alpha}{n}}}, \quad x \in \mathbf{R}^n,$$

where Q is a cube, $|Q|_\nu = \nu(Q)$, $|Q|$ denotes Lebesgue measure of Q. If $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n, d\mu)$, we set

$$M_\alpha(f d\mu)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| \, d\mu, \quad x \in \mathbf{R}^n.$$

In the case $0 < q < p$, [Verbitsky '92] showed, for $p > 1$,

$$M_\alpha \colon L^p(dx) \to L^q(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q}}(d\sigma),$$

$$M_\alpha \colon L^p(dx) \to L^{q,\infty}(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q},\infty}(d\sigma).$$

Two-weight maximal inequalities

More general two-weight maximal inequalities

$$\|M_\alpha(f d\mu)\|_{L^q(\sigma)} \le \varkappa \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu),$$

where characterized by [Sawyer '82] in the case $p = q > 1$, [Wheeden '93] in the case $q > p > 1$, and [Verbitsky '92] in the case $0 < q < p$ and $p > 1$, along with their weak-type counterparts,

$$\|M_\alpha(f d\mu)\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu),$$

where $\sigma, \mu \in \mathcal{M}^+(R^n)$, and $\varkappa, \varkappa_w$ are positive constants which do not dependent on f.

Some of the methods used for $0 < q < p$ and $p > 1$ are not applicable in the case $p = 1$, although there are analogues of these results for real Hardy spaces, when the norm $\|f\|_{L^p(\mu)}$ on the right-hand side is replaced with $\|M_\mu f\|_{L^p(\mu)}$,

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{|Q|_\mu} \int_Q |f| d\mu.$$

$(1, q)$-fractional maximal inequalities

We would like to understand similar problems in the case $0 < q < 1$ and $p = 1$, in particular, when $M_\alpha\colon \mathcal{M}^+(R^n) \to L^q(d\sigma)$, or equivalently, there exists a constant $\varkappa > 0$ such that the inequality

$$\|M_\alpha \nu\|_{L^q(\sigma)} \le \varkappa \, \|\nu\| \tag{4}$$

holds for all $\nu \in \mathcal{M}^+(R^n)$.

In the case $\alpha = 0$, [Rozin '77] showed that the condition

$$\sigma \in L^{\frac{1}{1-q},1}(R^n, dx)$$

is sufficient for the Hardy-Littlewood operator $M = M_0\colon L^1(dx) \to L^q(\sigma)$ to be bounded; moreover, when σ is radially symmetric and decreasing, this is also a necessary.

We generalize this result and provide necessary and sufficient conditions for the range $0 \le \alpha < n$. We also obtain analogous results for the weak-type inequality

$$\|M_\alpha \nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|\nu\|, \quad \text{for all } \nu \in \mathcal{M}^+(R^n). \tag{5}$$

We also treat more general dyadic maximal operators.

Strong-type fractional maximal inequality

Theorem

Let $\sigma \in M^+(R^n)$, $0 < q < 1$, *and* $0 \le \alpha < n$. *The inequality (4) holds if and only if there exists a function* $u \not\equiv 0$ *such that*

$$u \in L^q(\sigma), \quad \text{and} \quad u \ge M_\alpha(u^q \sigma).$$

Moreover, u *can be constructed as follows:* $u = \lim_{j \to \infty} u_j$, *where* $u_0 = (M_\alpha \sigma)^{\frac{1}{1-q}}$, $u_{j+1} \ge u_j$, *and*

$$u_{j+1} = M_\alpha(u_j^q \sigma), \quad j = 0, 1, \ldots.$$

In particular, $u \ge (M_\alpha \sigma)^{\frac{1}{1-q}}$.

Weak-type fractional maximal inequalities

For $\mathbf{0} \le \alpha < \mathbf{n}$, we define the *Hausdorff content* on a set $\mathbf{E} \subset \mathbf{R^n}$ to be

$$\mathbf{H}^{\mathbf{n}-\alpha}(\mathbf{E}) = \inf \left\{ \sum_{i=1}^{\infty} r_i^{n-\alpha} : \mathbf{E} \subset \bigcup_{i=1}^{\infty} \mathbf{B}(x_i, r_i). \right\}$$

Theorem

Let $\sigma \in \mathbf{M}^+(\mathbf{R^n})$, $\mathbf{0} < \mathbf{q} < \mathbf{1}$, *and* $\mathbf{0} \le \alpha < \mathbf{n}$. *Then TFAE:*

1. *There exists a positive constant* $\varkappa_w$ *such that*

$$\|\mathbf{M}_\alpha \nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|\nu\| \quad \text{for all } \nu \in \mathcal{M}(\mathbf{R^n}).$$

2. *There exists a positive constant* $\mathbf{C} > \mathbf{0}$ *such that*

$$\sigma(\mathbf{E}) \le \mathbf{C}\,(\mathbf{H}^{\mathbf{n}-\alpha}(\mathbf{E}))^{\mathbf{q}} \quad \text{for all Borel sets } \mathbf{E} \subset \mathbf{R^n}.$$

3. $\mathbf{M}_\alpha \sigma \in \mathbf{L}^{\frac{q}{1-q},\infty}(\sigma)$.

Remark. If $\alpha = \mathbf{0}$ each of the conditions (1)–$(3) \Leftrightarrow \sigma \in \mathbf{L}^{\frac{q}{1-q},\infty}(dx)$.

Part III: Applications to (fractional) Schrödinger operators

We obtain global bilateral bounds for Green's functions and kernels of Neumann series for a broad class of differential and integral equations with possibly singular coefficients, data, and boundaries of the domains.

Of greatest interest is the Schrödinger operator with potential $\mathbf{q}$, defined by $\mathbf{H} = -\mathbf{\Delta} - \mathbf{q}$, on a domain $\mathbf{\Omega} \subseteq \mathbf{R^n}$, for $\mathbf{n} \geq \mathbf{3}$. More generally, we consider the non-local operator $\mathbf{H}_\alpha = (-\mathbf{\Delta})^{\alpha/2} - \mathbf{q}$ and the associated Green's function on $\mathbf{\Omega}$. Here $\mathbf{q}$ and φ are locally integrable functions or locally finite measures. We allow $\mathbf{0} < \alpha < \mathbf{n}$ if $\mathbf{\Omega}$ is the entire space $\mathbf{R^n}$, and $\mathbf{0} < \alpha \leq \mathbf{2}$ for a bounded domain $\mathbf{\Omega}$, with $\mathbf{n} \neq \mathbf{2}$ if $\alpha = \mathbf{2}$.

We consider domains $\mathbf{\Omega}$ with a Green's operator $\mathbf{G}^{(\alpha)}$ for $(-\mathbf{\Delta})^{\alpha/2}$. Our theory is applicable to any bounded domain $\mathbf{\Omega}$ with the uniform Harnack boundary principle, established originally by Jerison and Kenig for NTA domains. This principle holds for a large class of domains in $\mathbf{R^n}$, $\mathbf{n} \geq \mathbf{2}$, including uniform domains in the classical case $\alpha = \mathbf{2}$, and even more general domains with the interior corkscrew condition if $\mathbf{0} < \alpha < \mathbf{2}$.

Main Theorem, linear case

Theorem (Frazier-Nazarov-Verbitsky '14)

If $||\mathbf{T}|| < \mathbf{1}$, *then*

$$\mathbf{G}^{(\alpha)}(\mathbf{x},\mathbf{y})e^{c_1 \Phi(\mathbf{x},\mathbf{y})} \le \mathbf{V}(\mathbf{x},\mathbf{y}) \le \mathbf{G}^{(\alpha)}(\mathbf{x},\mathbf{y})e^{c_2 \Phi(\mathbf{x},\mathbf{y})},$$

$$\Phi(\mathbf{x},\mathbf{y}) = \frac{1}{\mathbf{G}^{\alpha}(\mathbf{x},\mathbf{y})} \int_{\Omega} \mathbf{G}^{\alpha}(\mathbf{x},\mathbf{z})\, \mathbf{G}^{\alpha}(\mathbf{z},\mathbf{y})\, \mathbf{q}(\mathbf{z})\, d\mathbf{z}.$$

Global bilateral estimates of the conditional gauge:

$$e^{\Phi(\mathbf{x},\mathbf{y})} \le \mathbb{E}^{\mathbf{x}}_{\mathbf{y}} \left[e^{\int_0^{\zeta} \mathbf{q}(\mathbf{X}_s)\, ds} \right] \le e^{c\, \Phi(\mathbf{x},\mathbf{y})}.$$

$\mathbf{X}_t$ is a $\mathbf{y}$-conditioned properly scaled Brownian motion if $\alpha = \mathbf{2}$, or an α-stable symmetric Lévy process if $\mathbf{0} < \alpha < \mathbf{2}$ (starts at $\mathbf{x}$ and stops at $\mathbf{y}$, random lifetime ζ). Lower bound follows via Kac-Feynman formalism.

Reduction to Sobolev-Poincaré inequalities

A relationship between $||\mathbf{T}||$ and the best constant in the inequality:

$$\int_{\Omega} u^2 \, d\sigma \leq \beta^2 \int_{\Omega} \left| (-\triangle)^{\alpha/4} u \right|^2 dx,$$

for all test functions $u \in C_0^\infty(\Omega)$.

Lemma

Suppose $\Omega = \mathbf{R}^n$, $n \geq 1$ and $0 < \alpha < n$, or $\Omega \subset \mathbf{R}^n$, $n \geq 2$, is an open set with a positive Green's function $G^\alpha(\cdot, \cdot)$ and $0 < \alpha \leq 2$. Then the least constant above: $\beta = ||\mathbf{T}||_{L^2(\Omega,\sigma) \to L^2(\Omega,\sigma)}$. In particular, for every compact set $E \subset \Omega$:

$$\sigma(E) \leq \beta^2 \, \mathrm{cap}_\alpha(E, \Omega).$$

Here $\mathrm{cap}_\alpha(E, \Omega) = \inf\{||u||^2_{L^{\alpha/2,2}(\Omega)} : \; u \geq 1 \text{ on } E, \; u \in C_0^\infty(\Omega)\}$.

Linear differential equations

Classical time-independent Schrödinger equations

Let $\mathbf{\Omega} \subseteq \mathbf{R^n}$ be a domain. We want to find conditions for the existence of a positive solution $\mathbf{u}$ to the Dirichlet problem:

$$-\mathbf{\Delta u} = \boldsymbol{\sigma}\, \mathbf{u} + \varphi \text{ on } \mathbf{\Omega}, \quad \mathbf{u} = \mathbf{g} \text{ on } \partial\mathbf{\Omega},$$

for general nonnegative $\boldsymbol{\sigma}, \varphi$, $\mathbf{g}$ (measurable functions, or possibly measures). If $\mathbf{\Omega} = \mathbf{R^n}$, $\mathbf{g} = \mathbf{0}$, $\mathbf{u}$ vanishes at infinity.
Similar results for the fractional Laplacian:

$$(-\mathbf{\Delta})^{\alpha/2}\mathbf{u} = \boldsymbol{\sigma}\, \mathbf{u} + \varphi \text{ on } \mathbf{\Omega}, \quad \mathbf{u} = \mathbf{g} \text{ on } \mathbf{\Omega^c},$$

Global pointwise estimates of solutions, harmonic measure, Martin's kernels, etc. on NTA domains.

Equations on bounded NTA domains

Suppose $\Omega \subset \mathbf{R}^n$ is a bounded domain with the boundary Harnack principle for $(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha \le 2$. Then Green's function $G^\alpha(x, y)$ is **quasi-metrically modifiable**: let $m(x) = \min\left(1, G^\alpha(x, x_0)\right)$, where x_0 is a fixed pole in Ω. (For smooth Ω, $m(x) \approx \mathrm{dist}(x, \partial\Omega)^{\frac{\alpha}{2}}$.) Then (conjectured by Kalton-Verbitsky; Ancona '99; W. Hansen '05)

$$K(x, y) = \frac{G^\alpha(x, y)}{m(x) \cdot m(y)}$$

is a quasimetric kernel (sharp form of the 3-G inequality).

Inductively define K_j for $j \le 1$ by letting $K_1 = K$ and, for $j \ge 2$,

$$K_j(x, y) = \int_\Omega K(x, z) K_{j-1}(z, y)\, d\tilde{\sigma}(z), \quad d\tilde{\sigma} = m^2 d\sigma.$$

Apply the Main Theorem to estimate the modified Green's function

$$\widetilde{V}(x, y) = \sum_{j=1}^{\infty} K_j(x, y).$$

Quasi-metrically modifiable kernels

The following theorem holds for all quasi-metrically modifiable kernels $\mathbf{K}$:

Theorem (Frazier-Nazarov-Verbitsky '14)

$$\mathbf{V(x, y) \geq K(x, y) e^{\frac{1}{16\kappa^2} K_2(x,y)/K(x,y)}}$$

for all $\mathbf{x, y \in \Omega}$. *Conversely, if* $||\mathbf{T}|| < \mathbf{1}$ *there exist* $\mathbf{c > 0}$ *such that*

$$\mathbf{V(x, y) \leq K(x, y) e^{c\, K_2(x,y)/K(x,y)}}.$$

This yields Green's function estimates for Schrödinger operators.
Under the smallness condition $\|\boldsymbol{\sigma}\|_{\mathbf{wb}} \leq \mathbf{c}(\boldsymbol{\kappa})$:

$$\|\boldsymbol{\sigma}\|_{\mathbf{wb}} = \sup \frac{\mathbf{1}}{|\mathbf{E}|_{\tilde{\sigma}}} \iint_{\mathbf{E \times E}} \widetilde{\mathbf{K}}(\mathbf{x, y})\, \mathbf{d}\tilde{\sigma}(\mathbf{x})\, \mathbf{d}\tilde{\sigma}(\mathbf{y}), \quad \mathbf{E \subset \Omega},$$

(equivalent to boundedness of $\mathbf{T}$) [Frazier-V., '10]; enough to use $\mathbf{E = B}$.

Let $\mathbf{\Omega} \subseteq \mathbf{R}^{\mathbf{n}}$, $\mathbf{n} \geq \mathbf{2}$, and let $\boldsymbol{\sigma}$ be a nonnegative Borel measure in $\mathbf{\Omega}$. If $\boldsymbol{\sigma} \in \mathbf{L}^{\mathbf{1}}_{\mathbf{loc}}(\mathbf{\Omega})$ we write $\mathbf{d}\boldsymbol{\sigma} = \boldsymbol{\sigma}\, \mathbf{dx}$. Let $\mathbf{0} < \mathbf{q} < \mathbf{1}$. There exists a positive $\mathbf{W}^{\mathbf{1,2}}_{\mathbf{0}}$-solution $\mathbf{u}$ to the Dirichlet problem:

$$\begin{cases} -\triangle \mathbf{u} - \boldsymbol{\sigma}\, \mathbf{u}^{\mathbf{q}} = \mathbf{0} & \text{in } \mathbf{\Omega}, \\ \mathbf{u} = \mathbf{0} & \text{in } \partial\mathbf{\Omega}, \end{cases} \tag{6}$$

if $\mathbf{u} \in \mathbf{W}^{\mathbf{1,2}}_{\mathbf{0}}(\mathbf{\Omega}) \cap \mathbf{L}^{\mathbf{q}}_{\mathbf{loc}}(\mathbf{\Omega}, \mathbf{d}\boldsymbol{\sigma})$, $\mathbf{u} \geq \mathbf{0}$, and

$$\int_{\mathbf{\Omega}} \nabla \mathbf{u} \cdot \nabla \phi\, \mathbf{dx} = \int_{\mathbf{\Omega}} \phi\, \mathbf{u}^{\mathbf{q}}\, \mathbf{d}\boldsymbol{\sigma}, \tag{7}$$

for all $\phi \in \mathbf{C}^{\infty}_{\mathbf{0}}(\mathbf{\Omega})$. Here $\mathbf{W}^{\mathbf{1,2}}_{\mathbf{0}}(\mathbf{\Omega})$ is the **homogeneous** Sobolev (Dirichlet) space, the closure of $\mathbf{C}^{\infty}_{\mathbf{0}}(\mathbf{\Omega})$ in the norm $||\nabla \mathbf{u}||_{\mathbf{L}^2(\mathbf{\Omega})}$.

Finite energy solutions

Let $\mathbf{n \geq 2}$, $\boldsymbol{\Omega}$ an arbitrary open set so that the Laplacian has a Green's function $\mathbf{G(\cdot,\cdot)}$ (if $\mathbf{n = 2}$, cap$(\boldsymbol{\Omega}^{\mathbf{c}}) \neq \mathbf{0}$).

If (6) has a positive solution then $\boldsymbol{\sigma}$ is necessarily absolutely continuous with respect to harmonic capacity cap$(\cdot)$: more precisely, for any compact $\mathbf{E} \Subset \boldsymbol{\Omega}$ we have $\boldsymbol{\sigma}(\mathbf{E}) \leq \mathbf{c}\,\text{cap}(\mathbf{E})^{\mathbf{q}}$.

We denote by $\mathbf{U \geq 0}$ the unique solution of the Dirichlet problem

$$\begin{cases} -\triangle \mathbf{U} = \boldsymbol{\sigma} & \text{in } \boldsymbol{\Omega}, \\ \mathbf{U} = \mathbf{0} & \text{on } \partial\boldsymbol{\Omega}. \end{cases} \tag{8}$$

In other words, $\mathbf{U}$ is the Green potential of $\boldsymbol{\sigma}$:

$$\mathbf{U(x)} = \int_{\boldsymbol{\Omega}} \mathbf{G(x,y)}\, \mathbf{d}\boldsymbol{\sigma}\mathbf{(y)}.$$

Existence and uniqueness of finite energy solutions

Theorem (Cao-Verbitsky '14)

Let $0 < q < 1$, $\Omega \subseteq \mathbb{R}^n$. *There exists a solution* $u \in W_0^{1,2}(\Omega)$ *to equation (6) if and only if*

$$\int_\Omega U^{\frac{1+q}{1-q}}\, d\sigma < +\infty. \tag{9}$$

Moreover, such a solution $u \in L^{1+q}(\Omega, \sigma)$ *and is unique.*

In the case $\Omega = \mathbb{R}^n$, $n \geq 3$, condition (9) becomes

$$\int_{\mathbb{R}^n} (I_2\sigma)^{\frac{1+q}{1-q}}\, d\sigma < +\infty, \tag{10}$$

where $I_2\sigma = |\cdot|^{2-n} \star \sigma$, is the Newtonian potential of σ.

A crucial integral inequality: finite energy solutions

It turns out that this problem is closely related to the **trace inequality** (in the **non-classical** "upper triangle" case $\mathbf{1+q<2}$):

$$\left(\int_\Omega |\phi|^{1+q}\, d\sigma\right)^{1/(1+q)} \le C\, ||\nabla \phi||_{L^2(\Omega, dx)}, \quad \phi \in C_0^\infty(\Omega).$$

There is a capacitary characterization [Mazy'ya-Netrusov '95]:

$$\int_0^{|\Omega|_\sigma} \left(\frac{t}{\varkappa(\sigma, t)}\right)^{\frac{1+q}{1-q}} dt < +\infty,$$

$\varkappa(\sigma, t) = \inf\{\, \mathrm{cap}(E) : \ |E|_\sigma \ge t\}$; equivalent to a non-capacitary condition (in the case $\Omega = \mathbf{R}^n$ see [Cascante-Ortega-V. '00]):

$$\int_\Omega U^{\frac{1+q}{1-q}}\, d\sigma < +\infty.$$

A special case

Corollary

Under the assumptions of the theorem, if $\mathbf{n \geq 3}$, *and*

$$\sigma \in \mathbf{L^p(\Omega)}, \quad \mathbf{p} \geq \frac{\mathbf{2n}}{\mathbf{n+2-q(n-2)}},$$

then there exists a positive $\mathbf{W_0^{1,2}(\Omega)}$*-solution* $\mathbf{u} \in \mathbf{L^{1+q}(\Omega, \sigma)}$ *to (6). The exponent* $\mathbf{p}$ *is sharp. If* $\mathbf{n = 2}$, $\mathbf{p > 1}$, *this holds for bounded* $\mathbf{\Omega}$.

For **bounded** domains $\mathbf{\Omega}$, this is due to [Boccardo-Orsina, '94].

Bounded solutions

The following theorem characterizes **bounded** solutions in $\mathbf{R^n}$.

Theorem (Brezis-Kamin '92)

Let $0 < q < 1$, $\sigma \in L^\infty_{loc}(\mathbf{R^n})$. *There exists a bounded solution to equation (6) in* $\mathbf{R^n}$ *such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if* $U \in L^\infty(\mathbf{R^n})$. *Moreover, such a solution is unique, and satisfies the global estimates:*

$$U(x)^{\frac{1}{1-q}} \le u(x) \le C\,U(x), \quad x \in \mathbf{R^n}. \tag{11}$$

Here $U = I_2\sigma$ is the Newtonian potential, and $u = I_2(u^q\sigma)$. Both the lower and the upper estimates in (11) are sharp in a sense.

We will give more precise (matching) bilateral estimates using new **nonlinear potentials** of Wolff type, along with $U^{\frac{1}{1-q}}$.

General weak solutions and supersolutions

We now consider general weak solutions to (6) in a bounded (smooth) domain $\mathbf{\Omega}$: for all $\phi \in \mathbf{C}_0^2(\mathbf{\Omega})$,

$$\int_{\mathbf{\Omega}} -\mathbf{u} \triangle \phi \, \mathbf{dx} = \int_{\mathbf{\Omega}} \mathbf{u}^{\mathbf{q}} \, \phi \, \mathbf{d}\sigma,$$

where $\mathbf{u} \in \mathbf{L}^1(\mathbf{\Omega}, \mathbf{dx}) \cap \mathbf{L}^{\mathbf{q}}(\mathbf{\Omega}, \delta_{\mathbf{\Omega}} \, \mathbf{d}\sigma)$, $\mathbf{u} \geq \mathbf{0}$, quasi-continuous in $\mathbf{\Omega}$. (Note that σ necessarily does not charge sets of capacity zero.) Equivalently, $\mathbf{u} \geq \mathbf{0}$ ($\mathbf{u} \not\equiv +\infty$) is a solution to the integral equation

$$\mathbf{u}(\mathbf{x}) = \int_{\mathbf{\Omega}} \mathbf{G}(\mathbf{x}, \mathbf{y}) \, \mathbf{u}^{\mathbf{q}}(\mathbf{y}) \, \mathbf{d}\sigma(\mathbf{y}).$$

In the case $\mathbf{\Omega} = \mathbf{R}^{\mathbf{n}}$: $\mathbf{G}(\mathbf{x}, \mathbf{y}) = \mathbf{c}_{\mathbf{n}} |\mathbf{x} - \mathbf{y}|^{\mathbf{2}-\mathbf{n}}$, and $\mathbf{u} = \mathbf{I}_{\mathbf{2}}(\mathbf{u}^{\mathbf{q}} \, \mathbf{d}\sigma)$. For arbitrary Green domains $\mathbf{\Omega}$: this is the definition of the solution.

Extension of the Brezis-Kamin theorem to unbounded solutions

Theorem (Cao-Verbitsky '14)

Let $\mathbf{0} < \mathbf{q} < \mathbf{1}$ *and let* $\boldsymbol{\sigma}$ *be a positive measure on* $\mathbf{R^n}$ *such that, for every compact set* $\mathbf{E} \subset \mathbf{R^n}$,

$$|\mathbf{E}|_{\boldsymbol{\sigma}} \leq \mathbf{C}\,\mathrm{cap}(\mathbf{E}). \tag{12}$$

Then there exists a solution $\mathbf{u} > \mathbf{0}$ *to (6) such that* $\liminf_{|\mathbf{x}| \to +\infty} \mathbf{u}(\mathbf{x}) = \mathbf{0}$, *and*

$$\mathbf{U}(\mathbf{x})^{\frac{1}{1-\mathbf{q}}} \leq \mathbf{u}(\mathbf{x}) \leq \mathbf{C}\left(\mathbf{U}(\mathbf{x}) + \mathbf{U}(\mathbf{x})^{\frac{1}{1-\mathbf{q}}}\right), \quad \mathbf{x} \in \mathbf{R^n}, \tag{13}$$

provided $\mathbf{U} \not\equiv +\infty$ *(otherwise there is no solution).*

Both estimates are sharp as in the Brezis-Kamin theorem. The lower estimate holds for any $\boldsymbol{\sigma} \geq \mathbf{0}$, without (12). Condition (12) is weaker than $\mathbf{U} \in \mathbf{L^\infty(R^n)}$, and incorporates unbounded solutions $\mathbf{u}$.

Weak solutions: the radial case

In the radial case σ depends only on $r = |x|$ in R^n, $n \geq 3$,

$$U(r) = c_n \left(\frac{1}{r^{n-2}} \int_0^r t^{n-1}\, d\sigma(t) + \int_r^\infty t\, d\sigma(t) \right).$$

Theorem

Let $0 < q < 1$*. Suppose* σ *is radial. Then (6) has a solution iff*

$$\int_0^1 \frac{t^{n-1} d\sigma(t)}{t^{(n-2)q}} < +\infty, \quad \text{and} \quad \int_1^{+\infty} t\, d\sigma(t) < +\infty.$$

Moreover, the minimal solution u *satisfies:*

$$u(x) \approx U(r)^{\frac{1}{1-q}} + \frac{1}{r^{n-2}} \left(\int_0^r \frac{t^{n-1} d\sigma(t)}{t^{(n-2)q}} \right)^{\frac{1}{1-q}}.$$

Weak solutions: a crucial integral inequality

The problem of the existence of weak solutions to (6) is closely related to the following integral **$(\mathbf{p}, \mathbf{q})$-inequality** in the **very non-classical** case $\mathbf{p = 1}$, $\mathbf{0 < q < 1}$: for all $\phi \in \mathbf{C_0^2(R^n)}$ such that $\phi \geq \mathbf{0}$, $\triangle\phi \leq \mathbf{0}$,

$$\left(\int_{\mathbf{R^n}} \phi^{\mathbf{q}} \, \mathbf{d}\sigma\right)^{\frac{1}{\mathbf{q}}} \leq \kappa \int_{\mathbf{R^n}} |\triangle\phi| \, \mathbf{dx}.$$

Equivalently, a weighted norm inequality for Newtonian potentials holds:

$$\left(\int_{\mathbf{R^n}} (\mathbf{I_2}\nu)^{\mathbf{q}} \, \mathbf{d}\sigma\right)^{\frac{1}{\mathbf{q}}} \leq \kappa \, \nu(\mathbf{R^n}), \tag{14}$$

for all finite Borel measures ν on $\mathbf{R^n}$. More generally, for $(-\triangle)^{\frac{\alpha}{2}}$, $\mathbf{0} < \alpha < \mathbf{n}$,

$$\left(\int_{\mathbf{R^n}} (\mathbf{I}_\alpha\nu)^{\mathbf{q}} \, \mathbf{d}\sigma\right)^{\frac{1}{\mathbf{q}}} \leq \kappa \, \nu(\mathbf{R^n}).$$

By κ we will denote the least constant in these inequalities.

Localized integral inequality

We will need a local version of the preceding inequality, where the measure $\sigma = \sigma_B$ is restricted to a ball $\mathbf{B}$ in $\mathbf{R}^n$:

$$\left(\int_B (I_\alpha \nu)^q \, d\sigma \right)^{\frac{1}{q}} \leq \kappa_B \, \nu(\mathbf{R}^n),$$

for all positive finite measures ν in $\mathbf{R}^n$. The least constants κ_B (or κ_Q) will be used to define a new **intrinsic** potential of Wolff type.

Main Theorem

Theorem (Cao-Verbitsky, 2014)

Suppose $\Omega = \mathbf{R}^n$, *and* $0 < q < 1$. *Then (6) has a nontrivial (super) solution* $\mathbf{u}$ *such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if the following condition holds:*

$$\int_1^{+\infty} \frac{|B(0,r)|_\sigma}{r^{n-2}} \frac{dr}{r} + \int_1^{+\infty} \frac{\left(\kappa_{B(0,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r} < +\infty. \tag{15}$$

Moreover, there exists a minimal solution $\mathbf{u}$ *to (6) satisfying:*

$$u(x) \approx \Big(I_2\sigma(x)\Big)^{\frac{1}{1-q}} + \int_0^{+\infty} \frac{\left(\kappa_{B(x,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r}. \tag{16}$$

The second term is the intrinsic Wolff potential $\mathbf{K\sigma(x)}$ associated with (6).

Existence of $W^{1,2}_{loc}$ solutions

For the existence of a solution $u \in W^{1,2}_{loc}(R^n)$, an additional local version of the condition for finite energy solutions is needed:

$$\int_B (I_2\sigma_B)^{\frac{1+q}{1-q}}\, d\sigma < \infty, \qquad (17)$$

for all balls B in R^n; here $\sigma_B = \sigma|_B$.

Theorem (Cao-Verbitsky, 2014)

Under the assumptions of the previous theorem, there exists a nontrivial weak (super) solution $u \in W^{1,2}_{loc}(R^n)$ *to (19) such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if* (17) *holds together with*

$$\int_1^{+\infty} \frac{|B(0,r)|_\sigma}{r^{n-2}} \frac{dr}{r} + \int_1^{+\infty} \frac{\left(\kappa_{B(0,r)}\right)^{\frac{q}{1-q}}}{r^{n-2}} \frac{dr}{r} < +\infty.$$

Moreover, global estimates (16) *hold for the minimal solution* u.

The $(1, q)$-integral inequality on $\mathbf{R^n}$

Theorem

Let $0 < q < 1$, $0 < \alpha < n$, *and let* σ *be a measure on* $\mathbf{R^n}$.
(1) The integral inequality $||I_\alpha \nu||_{L^q(\sigma)} \le C\,||\nu||$ *holds for every finite measure* ν *if and only if there exists a* **finite** *measure* ω *on* $\mathbf{R^n}$ *such that*

$$d\sigma = \frac{d\omega}{(I_\alpha \omega)^q}. \tag{18}$$

(2) A **necessary** *condition:* $I_\alpha \sigma \in L^{\frac{q}{1-q}}(d\sigma)$.
(3) A **sufficient** *condition:* $I_\alpha \sigma \in L^{\frac{q}{1-q},1}(d\sigma)$.
(4) A **weak-type** *inequality* $||I_\alpha \nu||_{L^{q,\infty}(\sigma)} \le C\,||\nu||$ *holds if and only if* $I_\alpha \sigma \in L^{\frac{q}{1-q},\infty}(d\sigma)$, *or equivalently, for all compact sets* $E \subset \mathbf{R^n}$,

$$|E|_\sigma \le C\,\mathrm{cap}(E)^q.$$

Part V: Quasilinear Equations

We introduce a class of weak solutions to the quasilinear equation

$$-\Delta_p u - \sigma u^q = 0. \tag{19}$$

Here $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian operator, $1 < p < +\infty$. An analogue of the sublinear case is $0 < q < p - 1$. We denote by U a positive solution to

$$-\Delta_p U = \sigma, \quad \liminf_{|x| \to +\infty} U(x) = 0.$$

By [Kilpeläinen-Malý, '94; Phuc-Verbitsky, '08], $U \approx \mathcal{W}_{1,p}\sigma$, where Wolff's potential $\mathcal{W}_{1,p}\sigma$ is defined by

$$\mathcal{W}_{1,p}\sigma(x) = \int_0^\infty \left(\frac{|B(x,r)|_\sigma}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r}, \quad x \in \mathbf{R}^n.$$

Finite energy solutions for the $\mathbf{p}$-Laplacian

Theorem

Let $1 < p < n$ *and* $0 < q < p - 1$*. There exists a solution* $u \in W_0^{1,p}(\mathbf{R}^n)$ *to equation (19) if and only if*

$$\int_{\mathbf{R}^n} (\mathcal{W}_{1,p}\sigma)^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < +\infty. \tag{20}$$

Moreover, such a solution $u \in L^{1+q}(\mathbf{R}^n, \sigma)$ *and is unique.*

There are no nontrivial solutions on $\mathbf{R}^n$ if $p \geq n$.

Pointwise estimates in terms of Wolff's potentials

Theorem

Let $1 < p < n$ *and* $0 < q < p - 1$. *Let* σ *be a positive measure on* $\mathbf{R}^n$ *such that, for every compact set* $E \subset \mathbf{R}^n$,

$$|E|_\sigma \leq C \, \mathrm{cap}_{1,p}(E).$$

Then there exists a positive solution u *to (19) such that* $\liminf_{|x| \to +\infty} u(x) = 0$, *and*

$$C_1 (\mathcal{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq C_2 \left(\mathcal{W}_{1,p}\sigma + (\mathcal{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \right),$$

provided $\mathcal{W}_{1,p}\sigma \not\equiv +\infty$. *Otherwise there are no nontrivial solutions.*

Pointwise estimates in the general case

For $x \in \mathbf{R}^n, R > 0$. Let κ_B be the least constant in the inequality

$$||\mathcal{W}_{1,p}\nu||_{L^q(d\sigma_B)} \leq \kappa_B \, ||\nu||^{\frac{1}{p-1}}, \qquad (21)$$

for all positive measures ν on $\mathbf{R}^n$, and $B = B(x, r)$. Then the following estimates hold for the minimal solution u:

$$u(x) \approx (\mathcal{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} + \int_0^\infty \frac{(\kappa_{B(x,r)})^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \frac{dr}{r}. \qquad (22)$$

Existence of weak (renormalized) solutions

Theorem

Let $1 < p < n$ *and* $0 < q < p - 1$. *Let* σ *be a positive locally finite measure on* $\mathbf{R}^n$. *Then there exists a nontrivial (super) solution* u *to (19) such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if the following two conditions hold:*

$$\int_1^\infty \left(\frac{|B(0,r)|_\sigma}{r^{n-p}}\right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty, \tag{23}$$

$$\int_1^\infty \frac{\left(\kappa_{B(0,r)}\right)^{\frac{q(p-1)}{p-1-q}}}{r^{n-p}} \frac{dr}{r} < \infty. \tag{24}$$

In this case there is a minimal solution satisfying global estimates (22). Otherwise there are no nontrivial solutions. (The same is true if $p \geq n$.*)*

Existence of $\mathbf{W^{1,p}_{\mathrm{loc}}}$ solutions

If we wish to find a solution $\mathbf{u}$ in $\mathbf{W^{1,p}_{\mathrm{loc}}(R^n)}$, then an additional local version of the condition for finite energy solutions is needed:

$$\int_{\mathbf{B}} (\mathcal{W}_{1,p}\sigma_{\mathbf{B}})^{\frac{(1+q)(p-1)}{p-1-q}} \, d\sigma < \infty, \tag{25}$$

for all balls $\mathbf{B}$ in $\mathbf{R^n}$.

Theorem

Under the assumptions of the previous theorem, there exists a weak solution $\mathbf{u} \in \mathbf{W^{1,p}_{\mathit{loc}}(R^n)}$ *to (19) such that* $\liminf_{|x|\to+\infty} u(x) = 0$ *if and only if conditions* (23), (24) *and* (25) *hold. Moreover, global estimates* (22) *hold for the minimal solution.*