Global estimates of solutions to nonlinear elliptic PDE and integral equations

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Abstract

We will give necessary and sufficient conditions for the existence of solutions, and provide sharp global pointwise estimates of solutions and supersolutions in terms of nonlinear potentials adapted to certain linear PDE problems.

We will also discuss global pointwise estimates of positive solutions for more general nonlinear elliptic PDE of the type $-\Delta u + \sigma\, u^q = \mu$ for all real $q \neq 0$, where σ, μ are given functions, or signed measures, on a Euclidean domain $\Omega \subseteq \mathbf{R}^n$, or a weighted Riemannian manifold $\mathbf{M}$, and the Laplacian on $(\mathbf{M}, \omega dm_0)$

$$\Delta = \mathrm{div}_\omega \cdot \nabla, \quad \mathrm{div}_\omega = \frac{1}{\omega} \circ \mathrm{div} \circ \omega.$$

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Weighted manifolds

Let $\mathbf{M}$ be a Riemannian manifold, and ω be a smooth positive function on $\mathbf{M}$. Consider the measure $\mathbf{m}$ on $\mathbf{M}$ given by $\mathbf{dm} = \omega \mathbf{dm_0}$ where $\mathbf{m_0}$ is the Riemannian measure of $\mathbf{M}$. The couple $(\mathbf{M}, \mathbf{m})$ is called a weighted manifold. Set

$$\mathrm{div}_\omega = \frac{1}{\omega} \circ \mathrm{div} \circ \omega.$$

Here div and ∇ are the divergence and the gradient operators of the Riemannian structure of $\mathbf{M}$. Then

$$\Delta = \mathrm{div}_\omega \cdot \nabla$$

is called the Laplace operator on $(\mathbf{M}, \mathbf{m})$. If $\omega = 1$ then Δ is the Laplace-Beltrami operator.

Green functions

For a general ω, $\mathbf{\Delta}$ is symmetric with respect to the measure $\mathbf{m}$.
Moreover, $\mathbf{\Delta}$ satisfies the chain rule like the Laplace-Beltrami operator.

For any open connected set $\mathbf{\Omega} \subseteq \mathbf{M}$, denote by $\mathbf{G}^{\mathbf{\Omega}}(\mathbf{x}, \mathbf{y})$ the infimum of all positive fundamental solutions of $\mathbf{\Delta}$ in $\mathbf{\Omega}$.

Then the following is true:

either $\mathbf{G}^{\mathbf{\Omega}}(\mathbf{x}, \mathbf{y}) \equiv +\infty$ or $\mathbf{G}^{\mathbf{\Omega}}(\mathbf{x}, \mathbf{y}) < +\infty$ for all $\mathbf{x} \neq \mathbf{y}$.

In the latter case we will say that $\mathbf{G}^{\mathbf{\Omega}}$ is non-trivial, and call $\mathbf{G}^{\mathbf{\Omega}}$ the **minimal Green function** (positive, symmetric) of $\mathbf{\Delta}$ in $\mathbf{\Omega}$.

Green potentials

If $\mathbf{G}^{\Omega}$ is the minimal Green function, then for any function $\mathbf{f} \in \mathbf{L}^{1}_{\mathbf{loc}}(\Omega, \mathbf{m})$ set

$$\mathbf{G}^{\Omega}\mathbf{f}(\mathbf{x}) = \int_{\Omega} \mathbf{G}^{\Omega}(\mathbf{x}, \mathbf{y})\, \mathbf{f}(\mathbf{y})\, \mathbf{dm}(\mathbf{y}).$$

For $\mathbf{f} \geq 0$, the integral is understood in Lebesgue sense.
For a signed $\mathbf{f}$,

$$\mathbf{G}^{\Omega}\mathbf{f}(\mathbf{x}) = \mathbf{G}^{\Omega}\mathbf{f}_{+}(\mathbf{x}) - \mathbf{G}^{\Omega}\mathbf{f}_{-}(\mathbf{x})$$

assuming at least one of the following:

$$\mathbf{G}^{\Omega}\mathbf{f}_{+}(\mathbf{x}) < +\infty, \quad \text{or} \quad \mathbf{G}^{\Omega}\mathbf{f}_{-}(\mathbf{x}) < +\infty.$$

Then $\mathbf{G}^{\Omega}\mathbf{f}(\mathbf{x})$ is said to be **well-defined**.

If Ω is relatively compact then $\mathbf{G}^{\Omega}$ is non-trivial, $\mathbf{G}^{\Omega}(\mathbf{x}, \cdot) \in \mathbf{L}^{1}(\Omega)$, $\mathbf{G}^{\Omega}\mathbf{f}$ is finite for any $\mathbf{f} \in \mathbf{L}^{\infty}(\Omega)$.

Semi-linear equations: Problem 1

We are concerned with the following two model semi-linear elliptic problems. Our main goal is "sharp" pointwise estimates of positive sub/super-solutions.

Problem 1. Let $\Omega \subset M$ be an open relatively compact connected subdomain of M. Given $V \in C(\overline{\Omega})$, $\mu \in C\left(\overline{\Omega}\right)$, $\nu \in C(\partial\Omega)$, $\mu, \nu \geq 0$, **assume** that there exists a non-negative solution u to the following semi-linear Dirichlet problem:

$$\begin{cases} -\Delta u + V\, u^q \geq \mu & \text{in } \Omega \\ u \geq \nu & \text{in } \partial\Omega, \end{cases} \tag{1}$$

where $q > 0$, and

$$\begin{cases} -\Delta u + V\, u^q \leq \mu & \text{in } \Omega \\ u \leq \nu & \text{in } \partial\Omega, \end{cases} \tag{2}$$

where $q < 0$.

For simplicity: $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ is a classical solution.

The auxiliary linear Dirichlet problem

We will compare $\mathbf{u}$ to the solution $\mathbf{h}$ of the following auxiliary linear Dirichlet problem:

$$\begin{cases} -\Delta \mathbf{h} = \mu & \text{in } \Omega, \\ \mathbf{h} = \nu & \text{in } \partial\Omega, \end{cases}$$

where $\mathbf{h} \geq 0$ is **super-harmonic** in Ω.
We will write

$$\mathbf{h} = \mathbf{P}^{\Omega}\nu + \mathbf{G}^{\Omega}\mu.$$

For smooth domains $\mathbf{P}^{\Omega}\nu$ and $\mathbf{G}^{\Omega}\mu$ are given by the Poisson and Green integrals respectively.

Semi-linear equations: Problem 2

Problem 2. Let Ω be a open connected subset of a weighted manifold M. Assume G^{Ω} is non-trivial. Given $V \in C(\Omega)$ and $\mu \in C(\Omega)$, $\mu \geq 0$, assume that there exists a solution $u \geq 0$ of

$$-\Delta u + V\, u^q \geq \mu \quad \text{in } \Omega, \tag{3}$$

where $q > 0$, or a solution $u > 0$ of

$$-\Delta u + V\, u^q \leq \mu \quad \text{in } \Omega, \tag{4}$$

if $q < 0$.

1. We assume here that $u \in C^2(\Omega)$ and compare u to $h = G^{\Omega}\mu$, the minimal positive solution of $-\Delta h = \mu$ (in most cases, $\mu \not\equiv 0$).
2. In both Problems 1 and 2, regularity assumptions can be relaxed: Inequalities (1)-(4) can be understood in the weak sense, assuming V, μ, ν are merely locally integrable (or locally finite measures).

Main results: Problem 1

Theorem I

Let $(\mathbf{M}, \mathbf{m})$ *be a weighted manifold,* $\mathbf{\Omega} \subseteq \mathbf{M}$ *an open relatively compact subdomain of* $\mathbf{M}$, $\partial\mathbf{\Omega}$ *regular,* $\mathbf{V} \in \mathbf{C}(\overline{\mathbf{\Omega}})$, $\mu \in \mathbf{C}\left(\overline{\mathbf{\Omega}}\right)$, $\nu \in \mathbf{C}(\partial\mathbf{\Omega})$, $\mu, \nu \geq \mathbf{0}$, μ *locally Hölder continuous, either* $\mu \not\equiv \mathbf{0}$ *or* $\nu \not\equiv \mathbf{0}$*, which ensures that* $\mathbf{h} > \mathbf{0}$ *in* $\mathbf{\Omega}$.

Suppose $\mathbf{u} \in \mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}\left(\overline{\mathbf{\Omega}}\right)$ *is a non-negative solution to (1) if* $\mathbf{q} > \mathbf{0}$*, or (2) if* $\mathbf{q} < \mathbf{0}$.

Then the following statements hold for all $\mathbf{x} \in \mathbf{\Omega}$.

(i) If $\mathbf{q} = \mathbf{1}$*, then*

$$\mathbf{u(x)} \geq \mathbf{h(x)} e^{-\frac{1}{\mathbf{h(x)}} \mathbf{G}^{\mathbf{\Omega}}(\mathbf{hV})(\mathbf{x})}. \tag{5}$$

Main results: Problem 1

Theorem I (statements (ii), (iii))

(ii) If $\mathbf{q > 1}$*, then* **necessarily** *the condition*

$$-(q-1)G^{\Omega}(h^q V)(x) < h(x) \tag{6}$$

holds in Ω*, and*

$$u(x) \geq h(x) \left[1 + (q-1)\frac{G^{\Omega}(h^q V)(x)}{h(x)}\right]^{-\frac{1}{q-1}}. \tag{7}$$

(iii) If $0 < q < 1$*, then*

$$u(x) \geq h(x) \left[1 - (1-q)\frac{G^{\Omega}(\chi_{\Omega^+} h^q V)(x)}{h(x)}\right]_+^{\frac{1}{1-q}}, \tag{8}$$

where $\Omega^+ = \{x \in \Omega : \; u(x) > 0\}$.

Main results: Problem 1

Theorem I (continuation)

(iv) If $\mathbf{q < 0}$ *and* $\mathbf{u > 0}$ *in* $\mathbf{\Omega}$ *then* **necessarily** *the condition*

$$\mathbf{(1-q)G^{\Omega}(h^{q}V)(x) < h(x),} \tag{9}$$

holds in $\mathbf{\Omega}$*, and*

$$\mathbf{u(x) \le h(x)\left[1-(1-q)\frac{G^{\Omega}(h^{q}V)(x)}{h(x)}\right]^{\frac{1}{1-q}},} \tag{10}$$

provided $\mathbf{G^{\Omega}(h^{q}V)(x)}$ *is well-defined.*

Main results: Problem 2

Theorem II

Let $\mathbf{M}$ *be a weighted manifold, and* $\mathbf{\Omega} \subseteq \mathbf{M}$ *an arbitrary open subdomain of* $\mathbf{M}$ *with a non-trivial Green function* $\mathbf{G}^{\mathbf{\Omega}}$.
Suppose $\mathbf{V} \in \mathbf{C}(\mathbf{\Omega})$, $\mu \geq \mathbf{0}$ *locally Hölder continuous.*
Let $\mathbf{u} \in \mathbf{C}^2(\mathbf{\Omega})$ *satisfy inequality (3) if* $\mathbf{q} > \mathbf{0}$, *or (4) if* $\mathbf{q} < \mathbf{0}$.
For $\mathbf{h} = \mathbf{G}^{\mathbf{\Omega}}\mu$, *let* $\mathbf{x} \in \mathbf{\Omega}$ *be a point such that* $\mathbf{0} < \mathbf{h}(\mathbf{x}) < \infty$, *and* $\mathbf{G}^{\mathbf{\Omega}^+}(\mathbf{h}^{\mathbf{q}}\mathbf{V})(\mathbf{x})$ *is well-defined.*

Then statements (i)–(iii) of the previous Theorem are valid.

Statement (iv) is valid under the additional assumption that $\mathbf{u}$ *vanishes on the infinite boundary* $\partial_\infty\mathbf{\Omega}$, *i.e.,* $\lim_{\mathbf{k}\to\infty} \mathbf{u}(\mathbf{y}_\mathbf{k}) = \mathbf{0}$ *for any sequence* $\{\mathbf{y}_\mathbf{k}\}$ *in* $\mathbf{\Omega}$ *that leaves any compact subset of* $\mathbf{\Omega}$.

Statement (iv) fails without the last hypothesis.

Non-linear integral equations with positive kernel

Let (Ω, m) be a measure space with σ-finite non-negative measure m. The next theorem gives an abstract existence result together with pointwise estimates of solutions $0 < u < +\infty$ a.e. for the integral equation (for $q \in \mathbf{R} \setminus \{0\}$):

$$u(x) + \int_\Omega K(x,y)\, u(y)^q\, V(y)\, dm(y) = h(x) \quad \text{in } \Omega. \tag{11}$$

Here $K : \Omega \times \Omega \to \bar{\mathbf{R}}_+ \cup \{+\infty\}$ a non-negative measurable **kernel**. (More generally: replace $V dm$ by a measure $d\omega$.) For a Borel measure μ,

$$K\mu(x) = \int_\Omega K(x,y)\, d\mu(y),$$

$Kf = K(f dm)$ for a non-negative measurable function f.

Non-linear integral equations with positive kernel

The non-linear integral equation

$$\mathbf{u} + \mathbf{K}(\mathbf{u}^{\mathbf{q}}\,\mathbf{V}) = \mathbf{h}, \quad \mathbf{u} \geq \mathbf{0} \text{ in } \boldsymbol{\Omega},$$

serves as an abstract version of the equation

$$-\boldsymbol{\Delta}\mathbf{u} + \mathbf{u}^{\mathbf{q}}\mathbf{V} = \boldsymbol{\mu}, \quad \mathbf{u} \geq \mathbf{0} \text{ in } \boldsymbol{\Omega}, \tag{12}$$

where $\mathbf{u}$ is a **generalized** (**moderate**) solution with zero boundary values. In this case, $\mathbf{K} = \mathbf{G}^{\boldsymbol{\Omega}}$ is the Green function of the Laplacian $\boldsymbol{\Delta}$, and $\mathbf{h} = \mathbf{G}^{\boldsymbol{\Omega}}\boldsymbol{\mu}$ is the Green potential of a measure $\boldsymbol{\mu}$ in $\boldsymbol{\Omega}$.
For bounded smooth domains $\boldsymbol{\Omega}$, and $\boldsymbol{\mu} \in \mathbf{L}^{\mathbf{1}}(\boldsymbol{\Omega}, \mathbf{d}_{\boldsymbol{\Omega}}\mathbf{dx})$ this coincides with the notion of a **very weak** solution.

Existence and estimates of solutions

Theorem III

Let (Ω, m) *be a measure space with* σ*-finite measure* m, K *be a non-negative kernel on* $\Omega \times \Omega$. *Let* $\mu \geq 0$ *be a measurable function, or measure in* Ω, *such that*

$$h = K\mu < \infty \quad dm\text{−a.e. in } \Omega.$$

Let V *be a measurable function on* Ω. *Then the following statements hold.*
(i) For $q > 1$, *and* $V \leq 0$, *suppose that the following condition holds,*

$$K(h^q |V|)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{q-1} h(x) \quad dm\text{−a.e. in } \Omega. \qquad (13)$$

Then (11) has a **minimal** *solution* u *such that*

$$h(x) \leq u(x) \leq \frac{q}{q-1} h(x) \quad \text{in } \Omega. \qquad (14)$$

Existence and estimates of solutions

Theorem (continuation)

(ii) For $\mathbf{q < 0}$ *and* $\mathbf{V \geq 0}$*, suppose that the following condition holds,*

$$\mathbf{K(h^q V)(x) \leq \left(1 - \frac{1}{q}\right)^q \frac{1}{1-q} h(x) \quad dm - a.e. \ in \ \Omega.} \qquad (15)$$

Then (11) has a **maximal** *solution* $\mathbf{u}$ *such that*

$$\mathbf{\frac{1}{1-\frac{1}{q}} h(x) \leq u(x) \leq h(x) \quad dm - a.e. \quad in \ \Omega.} \qquad (16)$$

Some (incomplete) references

1. **Liner case $\mathbf{q = 1}$** (Schrödinger equations): lower estimates of Green's functions on domains and manifolds for $\mathbf{V \geq 0}$ [Grigor'yan-Hansen, 2008]. For $\mathbf{V \leq 0}$, [Frazier-Verbitsky, 2010], [Frazier-Nazarov-Verbitsky, 2014] two-sided estimates, general quasi-metric kernels $\mathbf{K}$ and arbitrary $\mathbf{V}$. See also [Murata, 1986] for examples and pointwise estimates.
2. **Superlinear case $\mathbf{q > 1}$**: For $\mathbf{V \geq 0}$, [Brezis-Cabré, 1998] and [Kalton-Verbitsky, 1999] existence results, two-sided estimates of solutions, examples. For $\mathbf{V \leq 0}$ (equations with absorption), see [Dyn'kin, 2004], [Marcus-Véron, 2014].
3. **Sublinear case $\mathbf{0 < q < 1}$**: $\mathbf{V \geq 0}$, **bounded** solutions on $\mathbf{R^n}$ [Brezis-Kamin, 1992]; existence of weak solutions with sharp two-sided estimates of solutions [Cao-Verbitsky, 2015], [Quinn-Verbitsky, 2016].
4. **Negative exponents:** $\mathbf{q < 0}$, $\mathbf{V(x) = \pm\, d_\Omega(x)^{-\beta}}$ $(\beta > \mathbf{0})$ $\mathbf{d_\Omega(x) = \mathrm{dist}(x, \partial\Omega)}$ [Dupaigne-Ghergu-Radulescu, 2007].

The Doob transform

Given a positive $\mathbf{C^2}$ function $\mathbf{h}$ in $\mathbf{\Omega}$, consider the following differential operator

$$\mathbf{L^h} = \frac{\mathbf{1}}{\mathbf{h}} \circ \mathbf{\Delta} \circ \mathbf{h}$$

acting on $\mathbf{C^2(\Omega)}$. The operator $\mathbf{L^h}$ is called the Doob transform of $\mathbf{\Delta}$. Usually it is used for a harmonic function $\mathbf{h}$, but we will use $\mathbf{L^h}$ for super-harmonic $\mathbf{h}$ as well.

Notice that $\mathbf{L^h}$ can be written in the form

$$\mathbf{L^h v} = \mathbf{\Delta^h v} + \frac{\mathbf{\Delta h}}{\mathbf{h}}\mathbf{v}, \tag{17}$$

where $\mathbf{v} \in \mathbf{C^2(\Omega)}$ and $\mathbf{\Delta^h}$ is the $\mathbf{h}$-Laplacian defined by

$$\mathbf{\Delta^h v} = \frac{\mathbf{1}}{\mathbf{h^2}}\mathrm{div}_\omega(\mathbf{h^2 \nabla v}). \tag{18}$$

Note that $\mathbf{\Delta^h}$ is the Laplace operator for the measure $\mathbf{h^2 dm} = \mathbf{h^2}\omega\mathbf{dm_0}$.

Inequalities for $\mathbf{L^h v}$; $\mathbf{v} = \phi^{-1}\left(\frac{\mathbf{u}}{\mathbf{h}}\right)$, ϕ increasing

Lemma

Let $\mathbf{h}$ be a positive $\mathbf{C^2}$-function in $\mathbf{\Omega}$. Let $\mathbf{u}$ be a solution of

$$-\Delta \mathbf{u} + \mathbf{V}\mathbf{u}^{\mathbf{q}} \geq -\Delta \mathbf{h} \tag{19}$$

in $\mathbf{\Omega}$, where $\mathbf{V} \in \mathbf{C}(\mathbf{\Omega})$ and $\mathbf{q} \in \mathbf{R} \setminus \{0\}$. Let ϕ be a $\mathbf{C^2}$ function on an interval $\mathbf{I} \subset \mathbb{R}$ such that $\phi' > 0$ in $\mathbf{I}$. Assume $\frac{\mathbf{u}}{\mathbf{h}}(\mathbf{\Omega}) \subset \phi(\mathbf{I})$.
Then $\mathbf{v} = \phi^{-1}\left(\frac{\mathbf{u}}{\mathbf{h}}\right)$ satisfies the differential inequality:

$$-\mathbf{L^h v} + \mathbf{h^{q-1}V}\frac{\phi(\mathbf{v})^{\mathbf{q}}}{\phi'(\mathbf{v})} \geq \mathbf{L^h 1}\left(\frac{\phi(\mathbf{v}) - 1}{\phi'(\mathbf{v})} - \mathbf{v}\right) + \frac{\phi''(\mathbf{v})}{\phi'(\mathbf{v})}|\nabla \mathbf{v}|^2. \tag{20}$$

If in place of (19) we have

$$-\Delta \mathbf{u} + \mathbf{V}\mathbf{u}^{\mathbf{q}} \leq -\Delta \mathbf{h}, \tag{21}$$

then (20) holds with $\leq$ instead of $\geq$.

Proof of the lemma

Set $\tilde{\mathbf{u}} = \frac{\mathbf{u}}{\mathbf{h}}$, so that $\mathbf{L}^{\mathbf{h}}\tilde{\mathbf{u}} = \frac{1}{\mathbf{h}}\boldsymbol{\Delta}\mathbf{u}$. Divide both sides of (19) by $\mathbf{h}$:

$$-\mathbf{L}^{\mathbf{h}}\tilde{\mathbf{u}} + \mathbf{h}^{\mathbf{q}-1}\mathbf{V}\tilde{\mathbf{u}}^{\mathbf{q}} \geq -\mathbf{L}^{\mathbf{h}}\mathbf{1}. \tag{22}$$

By the chain rule, for any $\mathbf{v} \in \mathbf{C}^2(\boldsymbol{\Omega})$

$$\boldsymbol{\Delta}^{\mathbf{h}}\phi(\mathbf{v}) = \phi'(\mathbf{v})\boldsymbol{\Delta}^{\mathbf{h}}\mathbf{v} + \phi''(\mathbf{v})|\nabla\mathbf{v}|^2,$$

which implies by (17)

$$\begin{aligned}
\mathbf{L}^{\mathbf{h}}\phi(\mathbf{v}) &= \boldsymbol{\Delta}^{\mathbf{h}}\phi(\mathbf{v}) + \frac{\boldsymbol{\Delta}\mathbf{h}}{\mathbf{h}}\phi(\mathbf{v}) \\
&= \phi'(\mathbf{v})\boldsymbol{\Delta}^{\mathbf{h}}\mathbf{v} + \phi''(\mathbf{v})|\nabla\mathbf{v}|^2 + \frac{\boldsymbol{\Delta}\mathbf{h}}{\mathbf{h}}\phi(\mathbf{v}) \\
&= \phi'(\mathbf{v})(\boldsymbol{\Delta}^{\mathbf{h}}\mathbf{v} + \frac{\boldsymbol{\Delta}\mathbf{h}}{\mathbf{h}}\mathbf{v}) + \phi''(\mathbf{v})|\nabla\mathbf{v}|^2 + \frac{\boldsymbol{\Delta}\mathbf{h}}{\mathbf{h}}\left(\phi(\mathbf{v}) - \mathbf{v}\phi'(\mathbf{v})\right) \\
&= \phi'(\mathbf{v})\mathbf{L}^{\mathbf{h}}\mathbf{v} + \phi''(\mathbf{v})|\nabla\mathbf{v}|^2 + \frac{\boldsymbol{\Delta}\mathbf{h}}{\mathbf{h}}\left(\phi(\mathbf{v}) - \mathbf{v}\phi'(\mathbf{v})\right).
\end{aligned}$$

The end of the proof

Therefore,

$$-\mathbf{L^h v} = -\frac{\mathbf{L^h}\phi(\mathbf{v})}{\phi'(\mathbf{v})} + \frac{\phi''(\mathbf{v})}{\phi'(\mathbf{v})}|\nabla \mathbf{v}|^2 + \frac{\Delta \mathbf{h}}{\mathbf{h}}\left(\frac{\phi(\mathbf{v})}{\phi'(\mathbf{v})} - \mathbf{v}\right). \tag{23}$$

Choose $\mathbf{v}$ from $\tilde{\mathbf{u}} = \phi(\mathbf{v})$; then (22) gives the following estimate:

$$-\mathbf{L^h}\phi(\mathbf{v}) + \mathbf{h^{q-1}V}\,\phi(\mathbf{v})^{\mathbf{q}} \geq -\mathbf{L^h 1}.$$

Using this inequality, we can get rid of $\mathbf{L^h}\phi(\mathbf{v})$ in (23):

$$-\mathbf{L^h v} + \mathbf{h^{q-1}V}\frac{\phi(\mathbf{v})^{\mathbf{q}}}{\phi'(\mathbf{v})} \geq \mathbf{L^h 1}\left(\frac{\phi(\mathbf{v}) - \mathbf{1}}{\phi'(\mathbf{v})} - \mathbf{v}\right) + \frac{\phi''(\mathbf{v})}{\phi'(\mathbf{v})}|\nabla \mathbf{v}|^2.$$

This proves the desired inequality for $\mathbf{L^h v}$.

Solutions to the converse inequality with $\leq$ in place $\geq$ are treated in the same way. □

Inequalities for $\Delta\,(hv)$; $v = \phi^{-1}\left(\frac{u}{h}\right)$

ϕ increasing, convex; **h** super-harmonic

Corollary

Under the hypotheses of the Lemma, assume in addition $\Delta h \le 0$ *in* Ω *and* $0 \in I$.
(i) If in the interval I,

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \ge 0, \tag{24}$$

and u *satisfies* $-\Delta u + V u^q \ge -\Delta h$, *then the function* $v = \phi^{-1}\left(\frac{u}{h}\right)$ *satisfies the following inequality in* Ω:

$$-\Delta\,(hv) + h^q V \frac{\phi(v)^q}{\phi'(v)} \ge 0. \tag{25}$$

Inequalities for $\mathbf{\Delta\,(hv)}$; $\mathbf{v} = \phi^{-1}\left(\frac{\mathbf{u}}{\mathbf{h}}\right)$

ϕ increasing, concave; **h** super-harmonic

Corollary (continuation)

(ii) If ϕ is concave,

$$\phi(0) = 1, \quad \phi' > 0, \quad \phi'' \leq 0, \tag{26}$$

and $\mathbf{u}$ *satisfies* $-\Delta \mathbf{u} + \mathbf{V}\mathbf{u}^{\mathbf{q}} \leq -\Delta \mathbf{h}$, *then* $\mathbf{v}$ *satisfies*

$$-\Delta\,(\mathbf{hv}) + \mathbf{h}^{\mathbf{q}}\mathbf{V}\frac{\phi(\mathbf{v})^{\mathbf{q}}}{\phi'(\mathbf{v})} \leq 0. \tag{27}$$

Proof of the corollary

Proof.

To prove (i), notice that, for a convex ϕ such that $\phi' > 0$, $\phi(0) = 1$,

$$\frac{\phi(v) - 1}{\phi'(v)} - v \geq 0,$$

since the chord of the graph of the convex function ϕ between the points $(0, 1)$ and $(v, \phi(v))$ lies above the tangent line at $(v, \phi(v))$.
Using also that $L^h 1 = \frac{\Delta h}{h} \leq 0$, we obtain from the Lemma:

$$-L^h v + h^{q-1} V \frac{\phi(v)^q}{\phi'(v)} \geq 0,$$

which is equivalent to (25), since $\Delta(hv) = h\, L^h v$. The proof of (ii) is similar. □

A minimum principle for super-harmonic functions

Lemma

Suppose $\boldsymbol{\Omega} \subseteq \mathbf{M}$ *is open, and* $\mathbf{F}$ *is a superharmonic function in* $\boldsymbol{\Omega}$*. Suppose* $\mathbf{F} = \mathbf{F_1} + \mathbf{F_2}$ *where* $\liminf_{x \to z} \mathbf{F_1}(x) \geq 0$ *for every* $z \in \partial_\infty \boldsymbol{\Omega}$*, and* $\mathbf{F_2} \geq -\mathbf{P}$*, where* $\mathbf{P} = \mathbf{G}^{\boldsymbol{\Omega}} \mu$ *is a Green potential of a positive measure* μ *in* $\boldsymbol{\Omega}$*,* $\mathbf{P} \not\equiv +\infty$ *on every component of* $\boldsymbol{\Omega}$*. Then* $\mathbf{F} \geq 0$ *in* $\boldsymbol{\Omega}$*.*

Proof.

Indeed, the function $\mathbf{F} + \mathbf{P}$ is obviously superharmonic, and $\mathbf{F} + \mathbf{P} \geq \mathbf{F_1}$. Hence $\liminf_{x \to z} (\mathbf{F} + \mathbf{P})(x) \geq 0$ for $z \in \partial_\infty \boldsymbol{\Omega}$, and by the maximum principle $\mathbf{F} + \mathbf{P} \geq 0$ on $\boldsymbol{\Omega}$. Hence $\mathbf{F}$ is a super-harmonic majorant of $-\mathbf{P}$, whose least super-harmonic majorant must be zero, which yields $\mathbf{F} \geq 0$. □

Semi-linear problems in "nice" domains for $\inf_\Omega h > 0$

Lemma

Suppose Ω *is a relatively compact domain in* M *with smooth boundary. Suppose* $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $V \in C(\overline{\Omega})$, *and* μ, ν *are non-negative functions such that* $\nu \in C(\partial\Omega)$, *and* $\mu \in C(\overline{\Omega}) \cap C^\alpha(\Omega)$ *for some* $\alpha \in (0, 1]$. *Let*

$$h = P^\Omega \nu + G^\Omega \mu. \tag{28}$$

If $\inf_\Omega h > 0$, *then the following statements hold.*
(i) In the case $q > 0$, *if* $u > 0$ *is a solution of*

$$\begin{cases} -\Delta u + V u^q \geq \mu & \text{in } \Omega, \\ u \geq \nu & \text{in } \partial\Omega, \end{cases} \tag{29}$$

then statements (i)-(iii) of Theorem 1 are valid (lower bounds for u*).*

Semi-linear problems in "nice" domains for $\inf_\Omega h > 0$

Lemma (continuation)

(ii) In the case $q < 0$, if $u > 0$ is a solution of

$$\begin{cases} -\Delta u + V u^q \le \mu & \text{in } \Omega, \\ u \le \nu & \text{in } \partial\Omega, \end{cases} \tag{30}$$

then statement (iv) of Theorem 1 is valid (upper bounds for u).

Proof of the Lemma

By the hypotheses, $\mathbf{h} \in \mathbf{C^2(\Omega)}$, $-\mathbf{\Delta h} = \mu$, and $\mathbf{h} > \mathbf{0}$ in $\mathbf{\Omega}$. Choose the function ϕ in the Corollary to satisfy the equation

$$\phi'(\mathbf{v}) = \phi(\mathbf{v})^{\mathbf{q}}. \tag{31}$$

For $\mathbf{q} = \mathbf{1}$, this gives

$$\phi(\mathbf{v}) = \mathbf{e^v}, \quad \mathbf{v} \in \mathbb{R}, \tag{32}$$

while for $\mathbf{q} \neq \mathbf{1}$, we obtain

$$\phi(\mathbf{v}) = [(\mathbf{1} - \mathbf{q})\mathbf{v} + \mathbf{1}]^{\frac{1}{1-q}}, \quad \mathbf{v} \in \mathbf{I_q}, \tag{33}$$

where the domain $\mathbf{I_q}$ of ϕ is given by:

$$\mathbf{I_q} = \begin{cases} (-\frac{1}{1-q}, +\infty) & \text{if } \mathbf{q} < \mathbf{1}, \\ (-\infty, +\infty) & \text{if } \mathbf{q} = \mathbf{1}, \\ (-\infty, \frac{1}{q-1}) & \text{if } \mathbf{q} > \mathbf{1}. \end{cases} \tag{34}$$

Proof of the Lemma (continuation)

Note that in all cases $\phi\left(I_q\right) = (0, \infty)$. Also, we have

$$\phi'(v) = [(1-q)v+1]^{\frac{q}{1-q}}, \quad \phi''(v) = q[(1-q)v+1]^{\frac{2q-1}{1-q}}. \qquad (35)$$

Since $u = h\phi\left(v\right)$, all the estimates in the case $q > 0$ follow from:

$$v(x) \geq -\frac{1}{h(x)} G^{\Omega}(h^q V)(x) \text{ for all } x \in \Omega. \qquad (36)$$

For $q < 0$, we will have the opposite inequality.
Let us use the function hv expressed explicitly via u and h as follows:

$$hv = \begin{cases} \frac{1}{q-1} h\left(1 - (\frac{h}{u})^{q-1}\right) & \text{if } 1 < q < +\infty, \\ h \log(\frac{u}{h}) & \text{if } q = 1, \\ \frac{1}{1-q}\left(h^q u^{1-q} - h\right) & \text{if } -\infty < q < 1. \end{cases} \qquad (37)$$

Proof of the Lemma (continuation)

Since $\mathbf{u} > 0$, $\mathbf{h} > 0$ in Ω, we have $\frac{\mathbf{u}}{\mathbf{h}}(\Omega) \subset \phi(\mathbf{I_q}) = (0, \infty)$, and $\mathbf{hv} \in C^2(\Omega)$.

In the case $\mathbf{q} > 0$ the function ϕ is concave, increasing, and $\phi(0) = 1$.

We obtain from the Corollary,

$$-\Delta\,(\mathbf{hv}) + \mathbf{h^q V} \geq 0. \tag{38}$$

Since $\mathbf{u} \geq \nu > 0$ on $\partial\Omega$, and consequently $\inf_\Omega \mathbf{u} > 0$, we actually have $\mathbf{hv} \in C\left(\overline{\Omega}\right) \cap C^2(\Omega)$, and $\mathbf{hv} \geq 0$ on $\partial\Omega$, which implies (36).

In addition, if $\mathbf{q} > 1$, then $\mathbf{I_q} = (-\infty, \frac{1}{q-1})$, so that $\mathbf{v(x)} < \frac{1}{q-1}$.

This gives the **necessary** condition for the existence of $\mathbf{u}$:

$$-G^\Omega(h^q V)(x) < \frac{1}{q-1} h(x), \quad \text{for all } x \in \Omega.$$

Proof of the Lemma (continuation)

Similarly, in the case $\mathbf{q} < \mathbf{0}$, we have $\mathbf{hv} \in \mathbf{C}\left(\overline{\boldsymbol{\Omega}}\right) \cap \mathbf{C}^2(\boldsymbol{\Omega})$ since $\inf_{\boldsymbol{\Omega}} \mathbf{h} > \mathbf{0}$. The inequality $\mathbf{u} \leq \boldsymbol{\nu}$ on $\partial\boldsymbol{\Omega}$ yields the boundary condition $\mathbf{hv} \leq \mathbf{0}$ on $\partial\boldsymbol{\Omega}$. By the Corollary 3 we obtain that in $\boldsymbol{\Omega}$,

$$-\boldsymbol{\Delta}\,(\mathbf{hv}) + \mathbf{h}^{\mathbf{q}}\mathbf{V} \leq \mathbf{0}, \quad \text{for all } \mathbf{x} \in \boldsymbol{\Omega}. \tag{39}$$

Together with the boundary condition this yields

$$\mathbf{v}(\mathbf{x}) \leq -\frac{\mathbf{1}}{\mathbf{h}(\mathbf{x})}\mathbf{G}^{\boldsymbol{\Omega}}(\mathbf{h}^{\mathbf{q}}\mathbf{V})(\mathbf{x}). \tag{40}$$

Hence, the upper bound (10) for $\mathbf{u}$ holds. Since $\mathbf{I}_{\mathbf{q}} = \left(-\frac{\mathbf{1}}{\mathbf{1}-\mathbf{q}}, +\infty\right)$, in this case $\mathbf{v}(\mathbf{x}) > -\frac{\mathbf{1}}{\mathbf{1}-\mathbf{q}}$. It follows that the necessary condition for the existence of $\mathbf{u}$ holds in this case as well. □

Proof of Theorem 1

Suppose $\Omega \subset M$ is a relatively compact domain whose boundary is regular with respect to the Dirichlet problem. Let

$$h = P^{\Omega}\nu + G^{\Omega}\mu > 0 \quad \text{in } \Omega. \tag{41}$$

Since μ is uniformly bounded in Ω, we have

$$G^{\Omega}\mu \le (\sup_{\Omega} \mu)\, G^{\Omega}1,$$

and hence by the regularity of $\partial\Omega$,

$$\lim_{y\longrightarrow x} G^{\Omega}\mu(y) = \lim_{y\longrightarrow x} G^{\Omega}1(y) = 0, \quad \lim_{y\longrightarrow x} P^{\Omega}\nu(y) = \nu(x), \quad x \in \partial\Omega.$$

It follows $h \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$, $-\Delta h = \mu$, and

$$\lim_{y\longrightarrow x} h(y) = \lim_{y\longrightarrow x} u(y) = \nu(x), \quad x \in \partial\Omega.$$

Proof of Theorem 1(continuation)

For $\epsilon > 0$, set $\mathbf{u}_\epsilon = \mathbf{u} + \epsilon$, $\mathbf{h}_\epsilon = \mathbf{h} + \epsilon$, and define the function $\mathbf{v}_\epsilon$ via

$$\frac{\mathbf{u}_\epsilon}{\mathbf{h}_\epsilon} = \phi\left(\mathbf{v}_\epsilon\right),$$

where ϕ is chosen as in the proof of the previous Lemma 6. Note that $\mathbf{h}_\epsilon > \mathbf{0}$ is superharmonic in $\mathbf{\Omega}$, and $-\mathbf{\Delta h}_\epsilon = \mu$. Clearly,
$\mathbf{h}_\epsilon, \mathbf{u}_\epsilon, \mathbf{v}_\epsilon \in \mathbf{C}^2(\mathbf{\Omega}) \cap \mathbf{C}\left(\overline{\mathbf{\Omega}}\right)$.
Identity (23) applied to $\mathbf{h}_\epsilon, \mathbf{u}_\epsilon, \mathbf{v}_\epsilon$ in place of $\mathbf{h}, \mathbf{u}, \mathbf{v}$ gives

$$-\mathbf{\Delta}(\mathbf{h}_\epsilon \mathbf{v}_\epsilon) = \frac{-\mathbf{\Delta u}}{\phi'(\mathbf{v}_\epsilon)} + \frac{\phi''(\mathbf{v}_\epsilon)}{\phi'\left(\mathbf{v}_\epsilon\right)}|\nabla \mathbf{v}|^2 \mathbf{h}_\epsilon + \mathbf{\Delta h}\left(\frac{\phi(\mathbf{v}_\epsilon)}{\phi'\left(\mathbf{v}_\epsilon\right)} - \mathbf{v}_\epsilon\right),$$

where

$$\phi'(\mathbf{v}_\epsilon) = \phi(\mathbf{v}_\epsilon)^{\mathbf{q}} = \left(\frac{\mathbf{u}_\epsilon}{\mathbf{h}_\epsilon}\right)^{\mathbf{q}}.$$

Proof of Theorem 1

Suppose $\mathbf{q} > \mathbf{0}$ and $-\boldsymbol{\Delta}\mathbf{u} \geq -\mathbf{V}\mathbf{u}^{\mathbf{q}} + \mu$, $\mu = -\boldsymbol{\Delta}\mathbf{h}$. Hence,

$$-\boldsymbol{\Delta}(\mathbf{h}_\epsilon \mathbf{v}_\epsilon) \geq -\mathbf{h}_\epsilon^{\mathbf{q}} \left(\frac{\mathbf{u}}{\mathbf{u}_\epsilon}\right)^{\mathbf{q}} \mathbf{V} + \frac{\phi''(\mathbf{v}_\epsilon)}{\phi'(\mathbf{v}_\epsilon)} |\nabla \mathbf{v}|^2 \mathbf{h}_\epsilon + \boldsymbol{\Delta}\mathbf{h} \left(\frac{\phi(\mathbf{v}_\epsilon) - \mathbf{1}}{\phi'(\mathbf{v}_\epsilon)} - \mathbf{v}_\epsilon\right).$$

Drop the last two non-negative terms on the right:

$$-\boldsymbol{\Delta}(\mathbf{h}_\epsilon \mathbf{v}_\epsilon) + \mathbf{h}_\epsilon^{\mathbf{q}} \left(\frac{\mathbf{u}}{\mathbf{u}_\epsilon}\right)^{\mathbf{q}} \mathbf{V} \geq \mathbf{0}.$$

Hence, the function

$$\mathbf{h}_\epsilon \mathbf{v}_\epsilon + \mathbf{G}^{\Omega}\left(\mathbf{h}_\epsilon^{\mathbf{q}} \left(\frac{\mathbf{u}}{\mathbf{u}_\epsilon}\right)^{\mathbf{q}} \mathbf{V}\right)$$

is superharmonic in $\boldsymbol{\Omega}$, and has non-negative boundary values:

$$\mathbf{h}_\epsilon \mathbf{v}_\epsilon = (\nu + \epsilon)\phi^{-1}\left(\frac{\mathbf{u} + \epsilon}{\nu + \epsilon}\right) \geq (\nu + \epsilon)\phi^{-1}(\mathbf{1}) = \mathbf{0} \quad \text{on } \partial\boldsymbol{\Omega},$$

since $\mathbf{u} \geq \nu$ on $\partial\boldsymbol{\Omega}$, ϕ is increasing, and $\phi(\mathbf{0}) = \mathbf{1}$.

Proof of Theorem 1

Consequently, by the minimum principle,

$$\mathbf{h_\epsilon v_\epsilon \geq -G^\Omega \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right) \quad \text{in } \Omega.} \tag{42}$$

Since $\mathbf{u \leq u_\epsilon}$, this implies

$$\mathbf{h_\epsilon v_\epsilon \geq -G^\Omega \left(h_\epsilon^q V_+ \right),} \tag{43}$$

where, in the case $\mathbf{q > 1}$ we additionally have

$$\mathbf{-\frac{G^\Omega \left(h_\epsilon^q V_+ \right)}{h_\epsilon} \leq -\frac{G^\Omega \left(h_\epsilon^q \left(\frac{u}{u_\epsilon} \right)^q V \right)}{h_\epsilon} \leq v_\epsilon < \frac{1}{q-1}.} \tag{44}$$

Let us show that in the case $\mathbf{q \geq 1}$ actually $\mathbf{u > 0}$ in $\mathbf{\Omega}$. In terms of $\mathbf{u_\epsilon}$, estimate (43) gives, for $\mathbf{q \geq 1}$,

$$\mathbf{u_\epsilon \geq h_\epsilon(x) \phi \left(-\frac{G^\Omega \left(h_\epsilon^q V_+ \right)}{h_\epsilon} \right).} \tag{45}$$

Proof of Theorem 1

Clearly, $\mathbf{h}_\epsilon \downarrow \mathbf{h}$, where $\mathbf{h} > \mathbf{0}$ by (41). Passing to the limit as $\epsilon \longrightarrow \mathbf{0}$, we deduce by the dominated convergence theorem, for $\mathbf{q} \geq \mathbf{1}$,

$$\mathbf{u} \geq \mathbf{h}\phi\left(-\frac{\mathbf{G}^\Omega\left(\mathbf{h}^\mathbf{q}\mathbf{V}_+\right)}{\mathbf{h}}\right) > \mathbf{0} \quad \text{in } \boldsymbol{\Omega}.$$

Note that here, for $\mathbf{q} > \mathbf{1}$, we have a strict inequality

$$-\frac{\mathbf{G}^\Omega\left(\mathbf{h}^\mathbf{q}\mathbf{V}_+\right)(\mathbf{x})}{\mathbf{h}(\mathbf{x})} < \frac{\mathbf{1}}{\mathbf{q}-\mathbf{1}},$$

since otherwise $\mathbf{u}(\mathbf{x}) = +\infty$.

Proof of Theorem 1

Hence, in the case $\mathbf{q} \geq \mathbf{1}$, we have $\mathbf{u} > \mathbf{0}$ in $\boldsymbol{\Omega}$. Consequently $\frac{\mathbf{u}}{\mathbf{u}_\epsilon} \uparrow \mathbf{1}$ as $\epsilon \downarrow \mathbf{0}$, and by the dominated convergence theorem,

$$\lim_{\epsilon \longrightarrow 0} \mathbf{G}^{\Omega}\left(\mathbf{h}_\epsilon^{\mathbf{q}}\left(\frac{\mathbf{u}}{\mathbf{u}_\epsilon}\right)^{\mathbf{q}}\mathbf{V}\right) = \mathbf{G}^{\Omega}\left(\mathbf{h}^{\mathbf{q}}\mathbf{V}\right). \tag{46}$$

The main estimate restated in terms of $\mathbf{u}_\epsilon$:

$$\mathbf{u}_\epsilon \geq \mathbf{h}_\epsilon(\mathbf{x})\phi\Big(- \frac{\mathbf{G}^{\Omega}\left(\mathbf{h}_\epsilon^{\mathbf{q}}\left(\frac{\mathbf{u}}{\mathbf{u}_\epsilon}\right)^{\mathbf{q}}\mathbf{V}\right)}{\mathbf{h}_\epsilon}\Big), \tag{47}$$

where by (44) the right-hand side is well-defined. Passing to the limit as $\epsilon \downarrow \mathbf{0}$, we deduce, for $\mathbf{q} \geq \mathbf{1}$,

$$\mathbf{u} \geq \mathbf{h}\phi\Big(- \frac{\mathbf{G}^{\Omega}\left(\mathbf{h}^{\mathbf{q}}\mathbf{V}\right)}{\mathbf{h}}\Big).$$

For $\mathbf{q} > \mathbf{1}$, additionally,

$$-\frac{\mathbf{G}^{\Omega}\left(\mathbf{h}^{\mathbf{q}}\mathbf{V}\right)}{\mathbf{h}} < \frac{\mathbf{1}}{\mathbf{q}-\mathbf{1}}.$$

Proof of Theorem 1

A similar argument applies for $\mathbf{0 < q < 1}$, but in this case $\mathbf{u}$ can be equal to zero on an open set, so that $\frac{\mathbf{u}}{\mathbf{u}_\epsilon} \uparrow \chi_{\Omega^+}$ as $\epsilon \downarrow \mathbf{0}$. Passing to the limit in (42) using the dominated convergence theorem as above gives

$$\mathbf{h v} \geq -\mathbf{G}^{\Omega} \left(\chi_{\Omega^+} \mathbf{h}^{\mathbf{q}} \mathbf{V} \right),$$

which is equivalent to the desired lower estimate for $\mathbf{u}$.
In the case $\mathbf{q < 0}$, we define the function $\mathbf{v}_\epsilon$ in a slightly different way, via the equation

$$\frac{\mathbf{u}}{\mathbf{h}_\epsilon} = \phi(\mathbf{v}_\epsilon),$$

where as before $\mathbf{h}_\epsilon = \mathbf{h} + \epsilon$, so that $-\boldsymbol{\Delta} \mathbf{h}_\epsilon = \mu$, and

$$\mathbf{h}_\epsilon \, \mathbf{v}_\epsilon = \frac{\mathbf{1}}{\mathbf{1 - q}} \, \mathbf{h}_\epsilon^{\mathbf{q}} \left(\mathbf{u}^{\mathbf{1-q}} - \mathbf{h}_\epsilon^{\mathbf{1-q}} \right) \in \mathbf{C}^2(\Omega) \cap \mathbf{C}(\overline{\Omega}). \qquad (48)$$

Proof of Theorem 1

Then

$$-\Delta(h_\epsilon v_\epsilon) + h_\epsilon^q V \le 0.$$

Since $u \le \nu$ on $\partial\Omega$, it follows

$$h_\epsilon v_\epsilon = \frac{1}{1-q}(\nu+\epsilon)^q\left(u^{1-q} - (\nu+\epsilon)^{1-q}\right) \le 0 \quad \text{on } \partial\Omega.$$

Hence,

$$h_\epsilon v_\epsilon \le -G^\Omega(h_\epsilon^q V) \quad \text{in } \Omega, \tag{49}$$

or, equivalently,

$$u \le h_\epsilon \left[1 - (1-q)\frac{G^\Omega(h_\epsilon^q V)}{h_\epsilon}\right]^{\frac{1}{1-q}} \quad \text{in } \Omega. \tag{50}$$

Proof of Theorem 1

From the above estimates we deduce

$$-\frac{\mathbf{G}^{\Omega}(\mathbf{h}_{\epsilon}^{\mathbf{q}}\mathbf{V})}{\mathbf{h}_{\epsilon}} \geq \mathbf{v}_{\epsilon} > -\frac{\mathbf{1}}{\mathbf{1}-\mathbf{q}}, \tag{51}$$

so that the expression in square brackets in is always positive. Moreover,

$$-\frac{\mathbf{G}^{\Omega}(\mathbf{h}_{\epsilon}^{\mathbf{q}}\mathbf{V}_{+})}{\mathbf{h}_{\epsilon}} + \frac{\mathbf{G}^{\Omega}(\mathbf{h}^{\mathbf{q}}\mathbf{V}_{-})}{\mathbf{h}} > -\frac{\mathbf{1}}{\mathbf{1}-\mathbf{q}}. \tag{52}$$

Since $\mathbf{q} < \mathbf{0}$, we have $\mathbf{h}_{\epsilon}^{\mathbf{q}} \uparrow \mathbf{h}^{\mathbf{q}}$ as $\epsilon \downarrow \mathbf{0}$. Using dominated convergence,

$$-\frac{\mathbf{G}^{\Omega}(\mathbf{h}^{\mathbf{q}}\mathbf{V})(\mathbf{x})}{\mathbf{h}(\mathbf{x})} \geq -\frac{\mathbf{1}}{\mathbf{1}-\mathbf{q}}. \tag{53}$$

Notice that here $\mathbf{G}^{\Omega}(\mathbf{h}^{\mathbf{q}}\mathbf{V}_{+})(\mathbf{x}) < +\infty$; otherwise

$$\mathbf{G}^{\Omega}(\mathbf{h}^{\mathbf{q}}\mathbf{V}_{\pm})(\mathbf{x}) = +\infty,$$

which contradicts the assumption that $\mathbf{G}^{\Omega}(\mathbf{h}^{\mathbf{q}}\mathbf{V})(\mathbf{x})$ is well-defined.

Proof of Theorem 1

Clearly, (50) yields at $\mathbf{x}$:

$$\mathbf{u} \leq \mathbf{h}_\epsilon \left[\mathbf{1} - (\mathbf{1} - \mathbf{q}) \, \frac{\mathbf{G}^\Omega(\mathbf{h}_\epsilon^\mathbf{q} \mathbf{V}_+)}{\mathbf{h}_\epsilon} + (\mathbf{1} - \mathbf{q}) \, \frac{\mathbf{G}^\Omega(\mathbf{h}^\mathbf{q} \mathbf{V}_-)}{\mathbf{h}} \right]^{\frac{1}{1-\mathbf{q}}} . \qquad (54)$$

By the dominated convergence theorem, we obtain the corresponding upper estimate at $\mathbf{x}$:

$$\mathbf{u}(\mathbf{x}) \leq \mathbf{h}(\mathbf{x}) \left[\mathbf{1} - (\mathbf{1} - \mathbf{q}) \, \frac{\mathbf{G}^\Omega(\mathbf{h}^\mathbf{q} \mathbf{V})(\mathbf{x})}{\mathbf{h}(\mathbf{x})} \right]^{\frac{1}{1-\mathbf{q}}} .$$

Since by assumption $\mathbf{u}(\mathbf{x}) > \mathbf{0}$, the expression in square brackets is strictly positive (necessary condition). □