Girsanov formula for Markov Jump Processes

1 Introduction

Good references but beyond the needs of the present course for Markov jump processes are chapter 9 of [3] and appendix 1 of [2]. Gardiner [1] in chapter 3 treats Markov jump processes as a special case of Markov processes and derives the master equation for their probability evolution. Before doing so here we will give as in [3] a brief description of *pathwise* realizations of jump processes.

2 Jump process

Let $\mathcal{S}$ a finite state space, and ξ_t is a jump Markov process

$$\xi_t \colon \mathbb{R}_+ \times \Omega \mapsto \mathcal{S} \tag{2.1}$$

The paths of a jump Markov process can be written (see e.g. [3]) as

$$\xi_t = \xi_o + \sum_{n=1}^{\infty} \zeta_n \, \mathbb{1}_{[0,t]}(T_n) \tag{2.2}$$

where

1. $\mathbb{1}$ denotes the characteristic function

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{2.3}$$

2. if $t = T_n$

$$\xi_{T_n} = \xi_0 + \sum_{i=1}^{n} \zeta_n \tag{2.4}$$

3. $\{T_n\}_{n=1}^{\infty}$ is a sequence of random variables with increments $\tau_n := T_{n+1} - T_n$, $T_0 = 0$ distributed if $\xi_{T_n} = \mathrm{x}$ according to

$$\tau_n \overset{d}{=} \mathrm{r}(\mathrm{x}) \, e^{-\mathrm{r}(\mathrm{x}) \, t} dt \tag{2.5}$$

4. if we posit $\xi_{T_n} = \mathrm{x}$ then

$$\zeta_{n+1} := \xi_{T_{n+1}} - \xi_{T_n} \equiv \xi_{T_{n+1}} - \mathrm{x} \tag{2.6}$$

takes the value

$$\zeta_{n+1} = \mathrm{x}' - \mathrm{x} \tag{2.7}$$

with probability depending only upon x

$$\zeta_{n+1} \stackrel{d}{=} \mathrm{p}(\mathrm{x}'|\mathrm{x}) = \left\{ \frac{\mathrm{k}(\mathrm{x}'|\mathrm{x})}{\mathrm{r}(\mathrm{x})} \right\}_{\mathrm{x}' \in \mathcal{S}} \tag{2.8}$$

5. We interpret the ζ_n's as jump amplitudes occurring at random times T_n. Self-consistency of the interpretation requires

$$\mathrm{P}(\mathrm{x}|\mathrm{x}) = 0 \tag{2.9}$$

6. The pair (ζ_n, T_n) is an inhomogeneous Markov chain on $\mathbb{S} \times \mathbb{R}_+$ with conditional probability

$$\mathrm{P}\left(\xi_{T_{n+1}} = \mathrm{x}', t' < T_{n+1} \leq t' + dt \,|\, \xi_{T_n} = \mathrm{x}, T_n = t\right) = \mathrm{p}\left(\mathrm{x}'|\mathrm{x}\right)\, \mathrm{r}\left(\mathrm{x}\right)\, e^{-\mathrm{r}(\mathrm{x})(t'-t)} H(t'-t) \tag{2.10}$$

The interpretation is as follows. The sequence of random increments $\{\Delta T_k\}_{k=0}^{\infty}$ paves the time horizon $[0, T_\infty] \subseteq \mathbb{R}_+$. Let ω be the event in the sample space Ω specifying a realization of the sequence $\{\Delta T_k(\omega)\}_{k=0}^{\infty}$. Let $0 \leq t \leq T_\infty$ be the time at which we observe the process. Then there is an $n_t \in \mathbb{N}$ such that

$$T_{n_t} := \sum_{n=0}^{n_t} \tau_k \;\leq\; t \;<\; \sum_{n=0}^{n_t+1} \tau_k := T_{n_t+1} \tag{2.11}$$

Such n_t counts the number of *jumps* occurring during the interval $[0, t] \subseteq [0, T_\infty]$. The size of each jump is specified by $\{Z_k\}_{k=0}^{n_*}$.

3 Poisson process as a special case of Markov jump process

A special case of (2.2) corresponds to the choice

$$\zeta_n = 1 \qquad \forall\, n \tag{3.1}$$

The general jump process reduces to the Poisson process starting from ξ_0:

$$\xi_t = \xi_0 + \sum_{n=1}^{\infty} \mathbb{1}_{[0,t]}(T_n) \;\in\; \xi_0 + \mathbb{N} \tag{3.2}$$

In particular, for $\xi_0 = 0$ and $t = T_n$ we have

$$\xi_{T_n} = n \tag{3.3}$$

We can compute the probability distribution using (2.5) and the independence of the jumps. To that goal let us first compute the probability density of

$$T_n = \sum_{i=1}^{n} \tau_i \tag{3.4}$$

This is most conveniently done by inverting the characteristic function

$$\mathrm{p}_{T_n}(t) = \int_{\mathbb{R}} \frac{dx}{2\,\pi}\, e^{-\imath\, x\, t}\, \mathrm{E}\, e^{\imath\, x\, T_n} = \int_{\mathbb{R}} \frac{dx}{2\,\pi}\, e^{-\imath\, x\, t}\, \left(\mathrm{E}\, e^{\imath\, x\, \tau}\right)^{n} \tag{3.5}$$

where

$$\mathrm{E}\, e^{\imath\, x\, \tau} = \int_0^\infty dt\, r\, e^{-r\, t + \imath\, t\, x} = \frac{r}{r - \imath\, x} \tag{3.6}$$

The anti-Fourier transform can be performed using Cauchy theorem

$$p_{T_n}(t) = \int_{\mathbb{R}} \frac{dx}{2\,\pi} \frac{r^n\, e^{\imath\, x\, t}}{(r - \imath\, t)^n} = \frac{r\,(r\, t)^{n-1}\, e^{-r\, t}}{\Gamma(n)} \tag{3.7}$$

In order to compare this result with the Poisson process as we defined it in the previous lecture, we observe that probability that the system has performed n jumps at time t is

$$\begin{aligned} &\mathrm{P}\,(\#_t = n) = \mathrm{P}\,(T_n \leq t, T_{n+1} > t) = \\ &\quad \int_0^t \mathrm{P}\,(s \,<\, T_n \leq s + ds, T_{n+1} > t) = \int_0^t ds\, \mathrm{P}\,(T_{n+1} > t\,|\, T_n = s)\ \mathrm{p}_{T_n}(s) \end{aligned} \tag{3.8}$$

We used here the short hand notation

$$\#_t := \text{number of jumps at time } t \tag{3.9}$$

Then we have

$$\mathrm{P}\,(T_{n+1} > t\,|\,T_n = s) = \mathrm{P}\,(\tau_{n+1} > t - T_n\,|\,T_n = s) = \int_{t-s}^\infty du\, r\, e^{-r\, u} \tag{3.10}$$

so that

$$\mathrm{P}\,(T_n \leq t, T_{n+1} > t) = \int_0^t ds\, e^{-r\,(t-s)} \frac{r\,(r\, s)^{n-1}\, e^{-r\, s}}{\Gamma(n)} \tag{3.11}$$

which allows us to recover

$$\mathrm{P}\,(\#_t = n) = \frac{(r\, t)^n\, e^{-r\, t}}{\Gamma(n+1)} \equiv \frac{(r\, t)^n\, e^{-r\, t}}{n!} \tag{3.12}$$

4 Averaging

Let F any bounded measurable function

$$F\colon\, (\mathbb{S} \times \mathbb{R}_+)^n \mapsto \mathbb{R} \tag{4.1}$$

Let also $\{t_i\}_{i=1}^n$ an ordered $\mathbb{R}_+$-valued n-tuple

$$t_1 \leq t_2 \leq t_3 \cdots \leq t_n = t \,<\, T_\infty \tag{4.2}$$

Suppose we need to evaluate the expectation value with respect to a Markov jump process $\{\xi_t\,, 0 \leq t \,<\, T_\infty\}$

$$\bar{F} := \mathrm{E} F(\xi_{t_1}\,, t_1\,, \dots\,, \xi_t\,, t) \tag{4.3}$$

Let us observe that

$$0 \leq t \leq T_\infty \qquad \Rightarrow \qquad \sum_{n=0}^\infty \mathbb{1}_{[T_n, T_{n+1})}(t) = 1 \tag{4.4}$$

Using (4.4) the expectation value reduces to the form

$$\bar{F} = \sum_{i=0}^{\infty} \bar{F}_i \tag{4.5a}$$

$$\bar{F}_i = \mathrm{E}F(\xi_{t_1}, t_1, \dots, \xi_t, t)\mathbb{1}_{[T_i, T_{i+1})}(t) \tag{4.5b}$$

Taking into account that $T_{n+1} = T_n + \tau_{n+1}$ and the mutual independence of the τ_n's we have the identity

$$\mathbb{1}_{[T_n, T_{n+1})}(t) = \mathbb{1}_{[0,t]}(T_n)\mathbb{1}_{(t-T_n,\infty)}(\tau_{n+1}) \tag{4.6}$$

whence we can write

$$\bar{F}_i = \mathrm{E}\left\{F(\xi_{t_1}, t_1, \dots, \xi_{T_i}, t)\, e^{-r(\xi_{T_i})\,(t-T_n)}\mathbb{1}_{[0,t)}(T_i)\right\} \tag{4.7}$$

The advantage of this writing is that each therm appearing in the series now contains only a finite number jumps, specifically i for $\bar{F}_i$. Note that we can now iterate the procedure for $\xi_{t_{n-1}}$ in order to finally arrive to an expression amenable to an elementary expression in terms of the transition probability (2.10)

5 Mean forward derivative of Markov jump process

We define the mean forward derivative of a Markov jump process as

$$D\xi_t := \lim_{dt\downarrow 0} \mathrm{E}_{\xi_t}\left\{\frac{\xi_{t+dt} - \xi_t}{dt}\right\} \tag{5.1}$$

Let us preliminarily observe that for any f depending upon the Markov jump process we can write any conditional expectation as the series

$$\mathrm{E}_{\xi_t} f(\cdot) = \sum_{i=0}^{\infty} \mathbb{1}_{[T_n, T_{n+1})}(t)\mathrm{E}_{\xi_{T_n}} f(\cdot) \tag{5.2}$$

Hence we need only to evaluate

$$\mathrm{E}_{\xi_{T_n}}\left\{(\xi_{t+dt} - \xi_{T_n})\right\} = \sum_{k=0}^{\infty} \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_k} - \xi_{T_n})\mathbb{1}_{[T_k, T_{k+1})}(t+dt)\right\} \tag{5.3}$$

Since $t + dt > t$ we can restrict the focus to addends satisfying $T_k \geq T_n$ and $k \geq n$:

$$\mathrm{E}_{\xi_{T_n}}(\xi_{t+dt} - \xi_{T_n}) = \sum_{k=n}^{\infty} \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_k} - \xi_{T_n})\mathbb{1}_{[T_k, T_{k+1})}(t+dt)\right\} \tag{5.4}$$

or equivalently

$$\mathrm{E}_{\xi_{T_n}}(\xi_{t+dt} - \xi_{T_n}) = \sum_{k=n}^{\infty} \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_k} - \xi_{T_n})\,\mathbb{1}_{[T_k, T_{k+1})}(t+dt)\right\} \tag{5.5}$$

As the addend for $k = n$ vanishes we only need to evaluate

$$X_{kn}(t) := \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_k} - \xi_{T_n})\,\mathbb{1}_{[T_k, T_{k+1})}(t+dt)\right\} \tag{5.6}$$

for $k > n$ and for $T_n \leq t < T_{n+1}$. By virtue of (4.6) we have

$$X_{kn}(t) := \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_k} - \xi_{T_n})\, e^{-r\left(\xi_{T_k}\right)(t+dt-T_k)}\mathbb{1}_{[0,t+dt]}(T_k)\,\mathbb{1}_{[T_n, T_{n+1})}(t)\right\} \tag{5.7}$$

We may distinguish two situations.

- $k = n+1$

$$X_{n+1n}(t) = \mathrm{E}_{\xi_{T_n}}\left\{\xi_{T_{n+1}}\, e^{-r\left(\xi_{T_{n+1}}\right)(t+dt-T_{n+1})}\, \mathbb{1}_{[0,t+dt)}(T_{n+1})\, \mathbb{1}_{[T_n,T_{n+1})}(t)\right\}$$
$$= \mathrm{E}_{\xi_{T_n}}\left\{\xi_{T_{n+1}}\, e^{-r\left(\xi_{T_{n+1}}\right)(t+dt-T_{n+1})}\, \mathbb{1}_{[0,t+dt]}(T_{n+1})\, \mathbb{1}_{[0,T_n]}(t)\, \mathbb{1}_{[t,\infty)}(T_{n+1})\right\} \tag{5.8}$$

The constraints imposed by the set characteristic functions yield

$$\mathbb{1}_{[0,t+dt]}(T_{n+1})\, \mathbb{1}_{[t,\infty)}(T_{n+1}) = \mathbb{1}_{[t,t+dt)}(T_{n+1}) = \mathbb{1}_{[t-T_n\,,t+dt-T_n)}(\tau_{n+1}) \tag{5.9}$$

We know explicitly, the probability density of the "time" increment variable

$$\tau_{n+1} \overset{d}{=} r\left(\xi_{T_n}\right)\, e^{-r(\xi_{T_n})\,t}\, H(t) \tag{5.10}$$

which yields

$$\int_{t-T_n}^{t+dt-T_n} ds\, e^{-r\left(\xi_{T_{n+1}}\right)(t+dt-T_n-s)} r\left(\xi_{T_n}\right)\, e^{-r(\xi_{T_n})\,s} = dt\, r\left(\xi_{T_n}\right)\, e^{-r(\xi_{T_n})\,(t-T_n)} + O(dt^2) \tag{5.11}$$

whence we obtain

$$X_{n+1n} = dt\, r\left(\xi_{T_n}\right)\, \mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_{n+1}} - \xi_{T_n})\, e^{-r(\xi_{T_n})\,(t-T_n)}\, \mathbb{1}_{[0,t]}(T_n)\right\} + O(dt^2) \tag{5.12}$$

The remaining average factorizes in

$$\mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_{n+1}} - T_n)\, e^{-r(\xi_{T_n})\,(t-T_n)}\, \mathbb{1}_{[0,t]}(T_n)\right\} =$$
$$\mathrm{E}_{\xi_{T_n}}\left\{(\xi_{T_{n+1}} - \xi_{T_n})\right\}\, \mathrm{E}_{\xi_{T_n}}\left\{e^{-r(\xi_{T_n})\,(t-T_n)}\, \mathbb{1}_{[0,t]}(T_n)\right\} \tag{5.13}$$

where

$$\mathrm{E}_{\xi_{T_n}}(\xi_{T_{n+1}} - \xi_{T_n}) = \sum_{\mathrm{x}\in\mathbb{S}} (\mathrm{x} - \xi_{T_n})\, \mathrm{p}\left(\mathrm{x}|\xi_{T_n}\right) \tag{5.14a}$$

$$\mathrm{E}_{\xi_{T_n}}\left\{e^{-r(\xi_{T_n})\,(t-T_n)}\, \mathbb{1}_{[0,t]}(T_n)\right\} = \mathrm{P}\left(\#_t = n|\xi_{T_n}\right) = \mathbb{1}_{[T_n,T_{n+1})}(t) \tag{5.14b}$$

We therefore proved that

$$X_{n-1n} = dt\, r\left(\xi_{T_{n-1}}\right)\, \mathbb{1}_{[T_{n-1},T_n)}(t) \sum_{\mathrm{x}\in\mathbb{S}} \mathrm{x}\, \mathrm{p}\left(\mathrm{x}|\xi_{T_{n-1}}\right) + O(dt^2) \tag{5.15}$$

2 If $k > n+1$. In such a case it is expedient to define

$$\tilde{T}_{k,n} = \sum_{l=n+2}^{k} \tau_l \tag{5.16}$$

so that

$$X_{kn}(t) = \mathrm{E}_{\xi_{T_n}}\left\{\xi_{T_k}\, e^{-r\left(\xi_{T_k}\right)(t+dt-T_{n+1}-\tilde{T}_{k,n})}\, \mathbb{1}_{[0,t+dt]}(T_{n+1}+\tilde{T}_{k,n})\mathbb{1}_{[T_n,T_{n+1})}(t)\right\}$$
$$= \mathrm{E}_{\xi_{T_n}}\left\{\xi_{T_k}\, e^{-r\left(\xi_{T_k}\right)(t+dt-T_{n+1}-\tilde{T}_{k,n})}\, \mathbb{1}_{[0,t+dt]}(T_{n+1}+\tilde{T}_{n,n})\mathbb{1}_{[0,t]}(T_n)\mathbb{1}_{[t,\infty]}(T_{n+1})\right\} \tag{5.17}$$

We observe

$$\mathbb{1}_{[0,t+dt]}(T_{n+1}+\tilde{T}_{k,n})\,\mathbb{1}_{[t,\infty]}(T_{n+1}) = \mathbb{1}_{[0,t+dt-T_{k,n}]}(T_{n+1})\,\mathbb{1}_{[t,\infty]}(T_{n+1})$$
$$= \mathbb{1}_{[0,dt]}(\tilde{T}_{k,n})\,\mathbb{1}_{[t\,,t+dt-T_{k,n}]}(T_{n+1}) = \mathbb{1}_{[0,dt]}(\tilde{T}_{k,n})\,\mathbb{1}_{[t-T_n\,,t+dt-T_n-T_{k,n}]}(\tau_{n+1}) \tag{5.18}$$

We can couch the last equality in to the form

$$X_{kn}(t) =$$
$$\mathrm{E}_{\xi_{T_n}}\left\{\xi_{T_k}\, e^{-r\left(\xi_{T_k}\right)(t+dt-T_n-\tau_{n+1}-\tilde{T}_{k,n})}\,\mathbb{1}_{[0,t]}(T_n)\,\mathbb{1}_{[0,dt]}(\tilde{T}_{k,n})\,\mathbb{1}_{[t-T_n\,,t+dt-T_n-T_{k,n}]}(\tau_{n+1})\right\} \tag{5.19}$$

whence we see that we are taking the expectation of a quantity containing the product of two characteristic functions of sets having support of linear size $O(dt)$. The conclusion is

$$X_{kn}(t) = O(dt^2) \tag{5.20}$$

Gleaning the above information we have shown that

$$D\xi_t = \lim_{dt\downarrow 0}\sum_{n=1}^{\infty}\frac{dt\, r\left(\xi_{T_{n-1}}\right)\ \mathbb{1}_{[T_{n-1},T_n)}(t)\ \sum_{\mathrm{x}\in\mathbb{S}}(\mathrm{x}-\xi_{T_n})\,\mathrm{p}\left(\mathrm{x}|\xi_{T_{n-1}}\right)+O(dt^2)}{dt} \tag{5.21}$$

or equivalently

$$D\xi_t = \sum_{\mathrm{x}\in\mathbb{S}}(\mathrm{x}-\xi_t)\,\mathrm{k}\,(\mathrm{x}|\xi_t) \tag{5.22}$$

6 Stochastic Equation satisfied by a Markov Jump process

The mean forward derivative suggests us that a Markov jump process may *pathwise* pathwise a stochastic equation of the form

$$d\xi_t = dt\, D\xi_t + d\mu_t \tag{6.1}$$

From the definition of mean forward derivative we must have

$$\mathrm{E}_{\xi_t} d\mu_t = 0 \tag{6.2}$$

for any t. If (6.1) holds true, then we can write

$$\xi_t - \xi_0 = \int_0^t dt\, D\xi_t + \int_0^t d\mu_t \tag{6.3}$$

As ξ_t is constant between jumps

$$\int_0^t dt\, D\xi_t = \sum_{n=0}^{1\infty} D\xi_{T_n}\,(t\wedge T_{n+1} - t\wedge T_n) \tag{6.4}$$

where

$$t_1\wedge t_2 = \begin{cases} t_1 & t_1\ \le\ t_2 \\ t_2 & t_1\ >\ t_2 \end{cases} \tag{6.5}$$

Note that in (6.4) when $t \leq T_n$

$$t \wedge T_{n+1} - t \wedge T_n = 0 \tag{6.6}$$

Thus we have for $\mu_0 = 0$

$$\mu_t = \xi_t - \xi_0 - \sum_{n=0}^{\infty} D\xi_{T_n} \left(t \wedge T_{n+1} - t \wedge T_n\right) \tag{6.7}$$

It is possible to prove [3] that if the probability of having very large jumps is "sufficiently" small the process exists μ_t and is a martingale.

7 Generator description

Let

$$f \colon \mathcal{S} \mapsto \mathbb{R} \tag{7.1}$$

a bounded, measurable function. We define the generator of a jump Markov process acting on f

$$(\mathsf{L}f)(\mathrm{x}) = \sum_{\mathrm{x}' \in \mathcal{S}} \left[f(\mathrm{x}') - f(\mathrm{x})\right] k(\mathrm{x}'|\mathrm{x}) = \sum_{\mathrm{x}' \in \mathcal{S}} f(\mathrm{x}')\, k(\mathrm{x}'|\mathrm{x}) - f(\mathrm{x})\, r(\mathrm{x}) \tag{7.2}$$

for $\mathrm{x}'\,, \mathrm{x} \in \mathcal{S}$, $k(\mathrm{x}'|\mathrm{x})$. The nullspace $\mathcal{N}(\mathfrak{L})$ of the generator $\mathfrak{L}$ consists in general of constant functions over $\mathcal{S}$:

$$\sum_{\mathrm{x}' \in \mathcal{S}} k(\mathrm{x}'|\mathrm{x})\, f - r(\mathrm{x})\, f = 0 \qquad \forall\, f \tag{7.3}$$

If the state of the system at time t is described by a probability distribution m

$$m \colon \mathcal{S} \times \mathbb{R}_+ \mapsto [0, 1] \tag{7.4}$$

we obtain

$$\frac{d}{dt}\mathrm{E}f = \sum_{\mathrm{x}, \mathrm{x}' \in \mathcal{S}} k(\mathrm{x}'|\mathrm{x}) \left[f(\mathrm{x}') - f(\mathrm{x})\right] m(\mathrm{x}, t) \tag{7.5}$$

If we then choose

$$f(\mathrm{x}) = \delta_{\mathrm{x},\mathrm{y}} \tag{7.6}$$

we recover the evolution equation for the measure

$$\frac{dm(\mathrm{y}, t)}{dt} = \sum_{\mathrm{x} \in \mathcal{S}} [k(\mathrm{y}|\mathrm{x})\, m(\mathrm{x}, t) - k(\mathrm{x}|\mathrm{y})\, m(\mathrm{y}, t)] = \sum_{\mathrm{x} \in \mathcal{S}} [k(\mathrm{y}|\mathrm{x}) \; - r\,(\mathrm{y}) \; \delta_{\mathrm{x}\,\mathrm{y}}]\, m(\mathrm{x}, t) \tag{7.7a}$$

$$m(\mathrm{y}, t_o) = m_o(\mathrm{y}) \tag{7.7b}$$

8 Girsanov formula: explicit expression of the Radon-Nikodym derivative

Let us consider two Markov chains $\Xi_1 = \{\boldsymbol{\xi}_{1;t}\,, t \in T\}$ and $\Xi_2 = \{\boldsymbol{\xi}_{2;t}\,, t \in T\}$ on the same countable space $\mathbb{S}$ with probability measures P_{Ξ_1} and P_{Ξ_2}. The probability measure P_{Ξ_1} is absolutely continuous with respect to P_{Ξ_2} up to a time t if the allowed jumps are the same. This means that for every $\boldsymbol{x} \in \mathbb{S}$ the sets

$$\{\boldsymbol{x} \in \mathbb{S} \,|\, \mathrm{p}_{\Xi_1}(\boldsymbol{x}|\boldsymbol{x}') \neq 0\} = \{\boldsymbol{x} \in \mathbb{S} \,|\, \mathrm{p}_{\Xi_2}(\boldsymbol{x}|\boldsymbol{x}') \neq 0\} \tag{8.1}$$

Proposition 8.1. *The Radon-Nikodym derivative restricted to $\mathcal{F}_t$ is given by the formula*

$$\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{\xi}_t) = \exp\left\{\int_0^t ds\;[r_1(\boldsymbol{\xi}_s) - r_2(\boldsymbol{\xi}_s)] - \sum_{s\leq t}\ln\frac{r_1(\boldsymbol{\xi}_{s^-})\,p_{\Xi_1}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})}{r_2(\boldsymbol{\xi}_{s^-})\,p_{\Xi_2}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})}\right\} \tag{8.2}$$

Proof. The assumption

$$\mathrm{p}_{\Xi_1}(\boldsymbol{x}|\boldsymbol{x}) = \mathrm{p}_{\Xi_2}(\boldsymbol{x}|\boldsymbol{x}) = 0 \tag{8.3}$$

ensures that

$$p_{\Xi_1}(\boldsymbol{\xi}_{2;s}|\boldsymbol{\xi}_{2;s^-}) = p_{\Xi_2}(\boldsymbol{\xi}_{2;s}|\boldsymbol{\xi}_{2;s^-}) = 0 \tag{8.4}$$

everywhere but at the jumps. In particular with probability one the sum $\sum_{s\leq t}$ reduces to a finite sum. To prove the claim we proceed in two steps.

i Let us fix an $n \in \mathbb{N}$ and

$$t = T_n = \sum_{i=1}^n \tau_i \tag{8.5}$$

For any bounded measurable function

$$F\colon (\mathbb{S}\times\mathbb{R}_+)^n \mapsto \mathbb{R} \tag{8.6}$$

we have

$$\begin{aligned}\mathrm{E}_{\mathrm{P}_{\Xi_1}} F\left(\boldsymbol{\xi}_{T_1}, T_1, \boldsymbol{\xi}_{T_2}, T_2, \dots, \boldsymbol{\xi}_{T_n}, T_n\right) &= \sum_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in\mathbb{S}}\prod_{0\leq i\leq n-1}\int_0^\infty ds_{i+1} \\ &\times p_{\Xi_1}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_1(\boldsymbol{x}_i)\,e^{-r_1(\boldsymbol{x}_i)\,s_{i+1}}\,F(\boldsymbol{x}_1, s_1, \boldsymbol{x}_2, s_1+s_2, \dots, \boldsymbol{x}_n, s_1+\dots+s_n)\end{aligned} \tag{8.7}$$

Dividing and multiplying by the measure of Ξ_2 we can couch the right hand side into the form

$$\begin{aligned}\mathrm{E}_{\mathrm{P}_{\Xi_1}} F\left(\boldsymbol{\xi}_{T_1}, T_1, \boldsymbol{\xi}_{T_2}, T_2, \dots, \boldsymbol{\xi}_{T_n}, T_n\right) &= \sum_{\boldsymbol{x}_1,\dots,\boldsymbol{x}_n\in\mathbb{S}}\prod_{0\leq i\leq n-1}\int_0^\infty ds_{i+1} \\ &\times p_{\Xi_2}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_2(\boldsymbol{x}_i)\,e^{-r_2(\boldsymbol{x}_i)\,s_{i+1}}\,\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{x}_{T_n})F(\boldsymbol{x}_1, s_1, \boldsymbol{x}_2, s_1+s_2, \dots, \boldsymbol{x}_n, s_1+\dots+s_n)\end{aligned} \tag{8.8a}$$

$$\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{x}_{T_n}) := \prod_{0\leq i\leq n-1}\frac{p_{\Xi_1}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_1(\boldsymbol{x}_i)}{p_{\Xi_2}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_2(\boldsymbol{x}_i)}\,e^{-[r_1(\boldsymbol{x}_i)-r_2(\boldsymbol{x}_i)]\,s_{i+1}} \tag{8.8b}$$

Since the factors are strictly positive definite we can also write

$$\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{x}_{2;t}) = \exp\left\{-\sum_{0\leq i\leq n-1}[r_1(\boldsymbol{x}_i)-r_2(\boldsymbol{x}_i)]\,s_{i+1}+\sum_{0\leq i\leq n-1}\ln\frac{p_{\Xi_1}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_1(\boldsymbol{x}_i)}{p_{\Xi_2}(\boldsymbol{x}_{i+1}|\boldsymbol{x}_i)\,r_2(\boldsymbol{x}_i)}\right\} \tag{8.9}$$

By definition $\tau_i = t_i$ and the process Ξ_2 is constant in between jumps. Hence

$$\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{\xi}_{T_n}) := \exp\left\{-\int_0^{T_n} ds\,[r_1(\boldsymbol{\xi}_s)-r_2(\boldsymbol{\xi}_s)]+\sum_{0\leq s\leq T_n}\ln\frac{p_{\Xi_1}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})\,r_1(\boldsymbol{\xi}_{s^-})}{p_{\Xi_2}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})\,r_2(\boldsymbol{\xi}_{s^-})}\right\} \tag{8.10}$$

and therefore

$$\begin{aligned}&\mathrm{E}_{\mathrm{P}_{\Xi_1}}F\left(\boldsymbol{\xi}_{\tau_1},\tau_1,\boldsymbol{\xi}_{\tau_1+\tau_2},\tau_1+\tau_2,\dots,\boldsymbol{\xi}_{T_n},T_n\right)=\\&\qquad\mathrm{E}_{\mathrm{P}_{\Xi_2}}\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{\xi}_{T_n})F\left(\boldsymbol{\xi}_{\tau_1},\tau_1,\boldsymbol{\xi}_{\tau_1+\tau_2},\tau_1+\tau_2,\dots,\boldsymbol{\xi}_{T_n},T_n\right)\end{aligned} \tag{8.11}$$

ii Let us now consider an ordered n-tuple $(t_1 \leq t_2 \leq \dots t_n = t)$. We can write

$$\begin{aligned}&\mathrm{E}_{\mathrm{P}_{\Xi_1}}F\left(\boldsymbol{\xi}_{t_1},t_1,\boldsymbol{\xi}_{t_2},t_2,\dots,\boldsymbol{\xi}_t,t\right)=\\&\qquad\sum_{n=0}^{\infty}\mathrm{E}_{\mathrm{P}_{\Xi_1}}F\left(\boldsymbol{\xi}_{t_1},t_1,\boldsymbol{\xi}_{t_2},t_2,\dots,\boldsymbol{\xi}_{T_n},t_n\right)\mathbb{1}_{[T_n,T_{n+1})}(t_n)\end{aligned} \tag{8.12}$$

Acting with (4.4) on all element of the n-tuple we arrive to an expression of the form

$$\begin{aligned}&\mathrm{E}_{\mathrm{P}_{\Xi_1}}F\left(\boldsymbol{\xi}_{t_1},t_1,\boldsymbol{\xi}_{t_2},t_2,\dots,\boldsymbol{\xi}_t,t\right)=\\&\qquad\sum_{\boldsymbol{n}}\mathrm{E}_{\mathrm{P}_{\Xi_1}}F_{\boldsymbol{n}}\left(\boldsymbol{\xi}_{T_1},t_1,\boldsymbol{\xi}_{T_2},t_2,\dots,\boldsymbol{\xi}_{T_n},t_n\right)e^{-r(\boldsymbol{\xi}_{T_n})(t-T_n)}\mathbb{1}_{[t_n,\infty)}(T_n)\end{aligned} \tag{8.13}$$

for some $\{F_{\boldsymbol{n}}\}$. On each term of the series we can now act as in step i. Since in the time interval $(T_n, t]$ no jump occurs, the only correction to the formula previously found comes from the exponential factor in (8.13). We can therefore write

$$\frac{d\mathrm{P}_{\Xi_1}}{d\mathrm{P}_{\Xi_2}}(\boldsymbol{\xi}_t) := \exp\left\{-\int_0^{t} ds\,[r_1(\boldsymbol{\xi}_s)-r_2(\boldsymbol{\xi}_s)]+\sum_{0\leq s\leq t}\ln\frac{p_{\Xi_1}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})\,r_1(\boldsymbol{\xi}_{s^-})}{p_{\Xi_2}(\boldsymbol{\xi}_s|\boldsymbol{\xi}_{s^-})\,r_2(\boldsymbol{\xi}_{s^-})}\right\} \tag{8.14}$$

□

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