Brownian motion

1 Introduction

Brownian motion is discussed in chapter 3 of [1].

2 Finite dimensional distributions of a stochastic process

Definition 2.1 (*Stochastic process*)**.** *A collection of random variables* $\{\xi_t | t \geq 0\}$

$$\boldsymbol{\xi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$$

is called a stochastic process.

Realizations of stochastic process are now paths rather than numbers:

Definition 2.2 (*Sample path*)**.** *For each* $\omega \in \Omega$ *the mapping*

$$t \to \boldsymbol{\xi}_t(\omega)$$

is called the sample path *of the stochastic process.*

In most applications stochastic processes are characterized by means of the family of all *finite dimensional joint distributions associated* to them. This means that for a stochastic process valued on $\mathbb{R}$, for any discrete sequence $\{t_i\}_{i=1}^n$ we consider the Borel σ-algebra $\mathcal{B}(\mathbb{R}^n)$ and $B_1, \ldots, B_n \, \mathcal{B}$ and consider

$$P_{\xi_t}(B_1, t_1, \ldots, B_n, t_n) \equiv P(\xi_{t_1} \in B_1, \ldots, \xi_{t_n} \in B_n)$$

The so defined families of joint probability yield a consistent description of a stochastic process

$$\boldsymbol{\xi} : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$$

if the following *Kolmogorov consistency conditions* are satisfied

i $P(\mathbb{R}^d, t) = 1$ for any t

ii $P_{\xi_t}(B_1, t_1, \ldots, B_n, t_n) \geq 0$

iii $P_{\xi_t}(B_1, t_1, \ldots, B_n, t_n) = P_{\xi_t}(B_1, t_1, \ldots, B_n, t_n, \mathbb{R}^d, t_{m+1})$

iv $P_{\xi_t}(B_{\pi(1)}, t_{\pi(1)}, \ldots, B_{\pi(n)}, t_{\pi(n)}) = P_{\xi_t}(B_1, t_1, \ldots, B_n, t_n, \mathbb{R}^d, t)$

3 Wiener process

The above definitions allow us to characterize the Wiener process as a stochastic process:

Definition 3.1 (*Wiener Process aka Brownian motion*)**.** *A real valued stochastic process*

$$w_t \,:\, \Omega \times \mathbb{R}_+ \,\to\, \mathbb{R}$$

is called a **Wiener process** *or* **Brownian motion** *if*

i $w_0 = 0$

ii any increment $w_t - w_s$ *has Gaussian probability density*

$$w_t - w_s \stackrel{d}{=} \frac{e^{-\frac{x^2}{2\,(t-s)}}}{\sqrt{2\,\pi\,(t-s)}} \tag{3.1}$$

for all $t \,\geq\, s \,\geq\, 0.$

iii For all times

$$t_1 \,<\, t_2 \,<\, \ldots \leq t_n$$

the random variables

$$w_{t_1}\,, w_{t_2} - w_{t_1}\,, \ldots\,, w_{t_n} - w_{t_{n-1}}$$

are independent *(the process has independent increments).*

3.1 Consequences of the definition

Some observations are in order

- It is not restrictive to consider the one dimensional case. A d-dimensional Wiener process a vector valued stochastic process whose components are each independent one-dimensional Wiener processes. More explicitly, the probability density of Brownian motion on $\mathbb{R}^d$ is given by

$$p_{\boldsymbol{w}_t}(\boldsymbol{x}) = \prod_{i=1}^{d} p_{w_t^i}(x_i)$$

- By i and ii we have that

$$p_{w_t}(x) = \frac{e^{-\frac{x^2}{2\,\sigma^2\,t}}}{(2\,\pi\,\sigma^2\,t)^{\frac{1}{2}}} \qquad \& \qquad p_{w_{t_2}-w_{t_1}}(x) = \frac{e^{-\frac{x^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2-t_1)]^{\frac{1}{2}}} \quad t_2 \,>\, t_1 \tag{3.2}$$

By iii The joint probability of w_{t_1} and $w_{t_2} - w_{t_1}$ is

$$p_{w_{t_1},w_{t_2}-w_{t_1}}(x_1, y) = \frac{e^{-\frac{x_1^2}{2\,\sigma^2\,t_1}}}{(2\,\pi\,\sigma^2\,t_1)^{\frac{1}{2}}} \frac{e^{-\frac{y^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2-t_1)]^{\frac{1}{2}}}$$

By definition of probability density we can also write

$$p_{w_{t_1},w_{t_2}-w_{t_1}}(x_1, y) = p_{w_{t_1},w_{t_2}-x_1}(x_1, y) = p_{w_{t_1},w_{t_2}}(x_1, y + x_1)$$

since

$$w_{t_2} = (w_{t_2} - w_{t_1}) + w_{t_1}$$

Recalling the definition of *conditional probability* we must also have

$$p_{w_{t_1},w_{t_2}}(x_1, y + x_1) = p_{w_{t_2}|w_{t_1}}(x_1 + y, t_2 \,|\, x_1, t_1)\, p_{w_{t_1}}(x_1) \quad \forall\, x_1\,, x_2\,, t_2\, >\, t_1$$

whence

$$p_{w_{t_2}|w_{t_1}}(x_1 + y, t_2 \,|\, x_1, t_1) = \frac{p_{w_{t_1},w_{t_2}}(x_1, y + x_1)}{p_{w_{t_1}}(x_1)} = \frac{p_{w_{t_1},w_{t_2}-w_{t_1}}(x_1, y)}{p_{w_{t_1}}(x_1)} = \frac{e^{-\frac{y^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2 - t_1)]^{\frac{1}{2}}}$$

Finally, upon setting $x_2 = y + x_1$ we get into:

$$p_{w_{t_2}|w_{t_1}}(x_2, t_2 \,|\, x_1, t_1) = \frac{e^{-\frac{(x_2-x_1)^2}{2\,\sigma^2\,(t_2-t_1)}}}{[2\,\pi\,\sigma^2\,(t_2 - t_1)]^{\frac{1}{2}}}$$

3.2 Continuity and non-differentiability of the Wiener process

Proposition 3.1. • *for each* $\gamma \in (1/2, 1]$ *and almost every realization* ω *(i.e. with* $\mathrm{P} = 1$*)*

$$w_t(\omega)\colon \mathbb{R}_+ \mapsto \mathbb{R} \tag{3.3}$$

is nowhere Hölder continuous with exponent γ*.*

- *The paths of Brownian motion are nowhere differentiable with probability one.*
For each 1 For each 1 ¡ 1 and almost every , t W(t,) is

Proof. Let aus fix $N \in \mathbb{N}$ large enough that

$$n\left(\gamma - \frac{1}{2}\right) > 1 \tag{3.4}$$

We say that a path w_t is Hölder continuous with exponent γ for som $s \infty I$ if

$$|w_t - w_s| \leq C\,|t - s|^{\gamma} \tag{3.5}$$

for all $t \in I$ and for some $C > 0$. Let us suppose $I = [0, 1]$. Let us now define the label

$$\lceil N\,s \rceil = \begin{cases} N\,s & \text{if} \quad n\,s \in \mathbb{N} \\ i & \text{if} \quad 1 > i - N\,s > 0 \end{cases} \tag{3.6}$$

for $N \in \mathbb{N}$ such that

$$N \gg n \tag{3.7}$$

The range of values

$$j = \lceil N\,s \rceil, \ldots, \lceil n\,s \rceil \cdots + n - 1 \tag{3.8}$$

taken by the integer label j are then such that

$$s \leq \frac{j}{N} \leq s + \frac{n}{N} \leq 1 \tag{3.9}$$

We have then that if (3.5) holds true

$$\begin{aligned} \left| w_{\frac{j+1}{N}} - w_{\frac{j}{N}} \right| &\leq \left| w_{\frac{j+1}{N}} - s \right| + \left| w_{\frac{j}{N}} - s \right| \\ &\leq C \left| \frac{j+1-N\,s}{N} \right|^{\gamma} + C \left| \frac{j-N\,s}{N} \right|^{\gamma} \leq \frac{2\,C\,n^{\gamma}}{N^{\gamma}} \end{aligned} \tag{3.10}$$

We can therefore pick some integer M such that

$$2\,C\,n^{\gamma} \leq M \tag{3.11}$$

for some integer M. Let us now define the event

$$F^{i}_{M,N} = \left\{ \omega \in \Omega | \left| w_{\frac{j+1}{N}} - w_{\frac{j}{N}} \right| \leq \frac{M}{N^{\gamma}}, \text{ for all } j = i, \ldots, i+n-1 \right\} \tag{3.12}$$

for which (3.10) holds true. The Wiener process is Hölder continuous at *some time* $s \in [0,1]$ if there exist an integer M and an integer N such that for any $k > N$ there exists some $i = 1, \ldots, N$ such that

$$\mathrm{P}(F^{i}_{M,N}) > 0 \tag{3.13}$$

The formal expression of the aforementioned condition is

$$F = \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \bigcup_{i=1}^{N} F^{i}_{M,N} \tag{3.14}$$

In particular if we show that

$$\bar{F}_{M,N} = \bigcap_{k=N}^{\infty} \bigcup_{i=1}^{N} F^{i}_{M,N} \tag{3.15}$$

has probability zero then we proved that the Wiener process is nowhere Hólder continuous with exponent γ. Since $\bar{F}_{M,N}$ is telescopic

$$\begin{aligned} \mathrm{P}\left(\bar{F}_{M,N}\right) &\leq \lim_{N \uparrow \infty} \inf_{N} \mathrm{P}(\bigcup_{i=1}^{N} \bar{F}^{i}_{M,N}) \\ &\leq \lim_{N \uparrow \infty} \sum_{i=1}^{n} \mathrm{P}(\bar{F}^{i}_{M,N}) \leq \lim_{N \uparrow \infty} N \left[\mathrm{P}\left(\left| w_{\frac{1}{N}} \right| \leq \frac{M}{N^{\gamma}} \right) \right]^{n} \end{aligned} \tag{3.16}$$

The last inequality follows because the Wiener product has independent and stationary increments

$$w_{\frac{j+1}{N}} - w_{\frac{j}{N}} \overset{d}{=} w_{\frac{1}{N}} \tag{3.17}$$

We can compute the probability

$$\mathrm{P}\left(\left|w_{\frac{1}{n}}\right| \leq \frac{M}{N^{\gamma}}\right) = \int_{-\frac{M}{N^{\gamma}}}^{\frac{M}{N^{\gamma}}} dx \frac{e^{-\frac{N\,x^2}{2}}}{\sqrt{2\,\pi\,N^{-1}}} \leq \tilde{K} N^{\frac{1}{2}-\gamma} \tag{3.18}$$

We see that it goes to zero for any $\gamma > 1/2$. Furthermore

$$\mathrm{P}\left(\bar{F}_{M,N}\right) \leq \tilde{K} \lim_{N\uparrow\infty} N^{1+n\left(\frac{1}{2}-\gamma\right)} = 0 \tag{3.19}$$

since we hypothesized (3.4). □

3.2.1 Observations

- In probability Hölder continuity can be argued by elementary considerations

$$\mathrm{P}\left(|w_t| > \epsilon\right) = 2\int_{\varepsilon}^{\infty} dx \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} \leq 2\int_{\varepsilon}^{\infty} dx \frac{x}{\varepsilon} \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sqrt{\frac{2\,t}{\pi}} \frac{e^{-\frac{\varepsilon^2}{2\,t}}}{\varepsilon} \tag{3.20}$$

  For any

$$\varepsilon = K\,t^{\gamma} \qquad\qquad K > 0 \tag{3.21}$$

  the probability tends to zero if $\gamma < 1/2$. The conclusion is that in probability

$$|w_t| \leq K\,t^{\gamma} \qquad\qquad 0 < \gamma < \frac{1}{2} \tag{3.22}$$

- An indication of non-differentiability comes from the evaluation of the expected value of the absolute value of the incremental ratio of the Wiener process:

$$\mathrm{E}\frac{|w_t|}{t} = 2\int_{0}^{\infty} dx \frac{x}{t} \frac{e^{-\frac{x^2}{2\,t}}}{\sqrt{2\,\pi\,t}} = \sqrt{\frac{2}{\pi\,t}} \tag{3.23}$$

  which diverges as t tends to zero.

- The probability density of

$$\eta_t = \frac{w_t}{t} \tag{3.24}$$

  is

$$\mathrm{p}(x) = \sqrt{\frac{t}{2\,\pi}}\, e^{-\frac{t\,x^2}{2}} \tag{3.25}$$

  whence for any $0 \leq L < \infty$

$$\lim_{t\downarrow 0} \mathrm{P}(-L \leq \eta \leq L) = \lim_{t\downarrow 0} \int_{-L}^{L} dx \sqrt{\frac{t}{2\,\pi}}\, e^{-\frac{t\,x^2}{2}} = 0 \tag{3.26}$$

  This fact indicates that the probability concentrates at infinity.

4 Markov processes and Chapman-Kolmogorov equation

A special important class of non-anticipating stochastic processes are Markov processes

Definition 4.1. *Let $\mathcal{F}^{\xi}_{[0,t]}$ the filtration generated by a stochastic process ξ_t. Then ξ_t is Markov if for any $t \geq s$ and any event A*

$$P\left(\xi_t \in A|\boldsymbol{\mathcal{F}}^{\xi}_{[0,s]}\right) = P\left(\xi_t \in A|\mathcal{F}^{\xi}_{s}\right) = P\left(\xi_t \in A|\xi_s\right) \tag{4.1}$$

This means that the state of the system at time s fully specify further evolution independently of what happened before for times smaller s. The system has no "memory". In particular if the transition probability p density of the Markov process $\xi_t : \Omega \times I \mapsto \mathbb{R}$ is available

$$P(\xi_t \in A|\xi_s = y) = \int_A dx\, p(x, t \,|\, y, s) \tag{4.2}$$

then the definition of Markov process implies that the Chapman-Kolmogorov equation

$$p(x_2, t_2 \,|\, x_0, t_0) = \int_{\mathbb{R}} dx_1\, p(x_2, t_2 \,|\, x_1, t_1)\, p(x_1, t_1 \,|\, x_0, t_0) \tag{4.3}$$

must hold true for any (x_2, x_0) and for any t_i, $i = 0, 1, 2$.

4.1 Quadratic variation of the Wiener process

In the case of the Wiener process we have

Proposition 4.1 (*Quadratic variation of the B.M.*)**.** *The quadratic variation of the Brownian motion in $[0, t]$ for any $t \in \mathbb{R}_+$*

$$V_{[w\,,w]}(I) \;= t$$

in the sense of $\mathbb{L}^2(\Omega)$.

Proof. By direct calculation we know that

$$\mathrm{E} w_t^2 = t \tag{4.4}$$

Let p_n a partition paving $[0, t]$ with n sub-intervals:

$$Q_n := \sum_{\mathsf{p}_n} (w_{t_k} - w_{t_{k-1}})^2$$

we have then

$$\mathrm{E}\,(Q_n - t)^2 = \mathrm{E} \sum_{k\, l \in \mathsf{p}_n} \left[(w_{t_k} - w_{t_{k-1}})^2 - (t_k - t_{k-1})\right] \left[(w_{t_l} - w_{t_{l-1}})^2 - (t_l - t_{l-1})\right]$$

For non-overlapping intervals, the averaged quantities are independent random variables with zero average. The only contributions to the sum come from overlapping intervals:

$$\mathrm{E}(Q_n - t)^2 = \mathrm{E} \sum_{k \in \mathsf{p}_n} \left[(w_{t_k} - w_{t_{k-1}})^2 - (t_k - t_{k-1})\right]^2 = 2 \sum_{k \in \mathsf{p}_n} (t_k - t_{k-1})^2$$

whence

$$\mathrm{E}(Q_n - t)^2 \leq 2\, t \max_{k \in \mathsf{p}_n} (t_k - t_{k-1}) \overset{\max_{k \in \mathsf{p}_n}(t_k - t_{k-1}) \downarrow 0}{\to} 0$$

□

The finite value of the quadratic variation motivates the estimate

$$dw_t \sim O(\sqrt{dt})$$

for typical increments of the Wiener process.

References

[1] L. C. Evans. An introduction to stochastic differential equations. UC Berkeley, Lecture Notes, 2006.