Lecture 12: Stochastic integrals

1 Introduction

Evans [1] discusses stochastic integral in § B amd § C of chapter IV. In § D [1] derives Ito lemma as a result for differentials along paths specified by non-anticipative functionals of the Wiener process. We showed in lecture 9 that Ito lemma can be regarded as a result in standard analysis for functions of paths with *finite second variation*. Chapter IV of [2] in § 4.1–4.2 also covers the construction of stochastic integrals.

2 Stochastic integrals

Let f an analytic function

$$f : \mathbb{R} \to \mathbb{R}$$

we would like to make sense of the functional of the Brownian motion

$$I = \int_0^t f(w_s)\, dw_s \qquad \text{mathematics notation} \tag{2.1}$$

sometimes also written as

$$I = \int_0^t f(w_s)\, \eta_s\, ds \qquad \text{physics notation} \tag{2.2}$$

Note that the physics notation implies that η_s is the "derivative" of the Wiener process which we showed to be nowhere differentiable. We should therefore interpret $\eta_s\, ds$ only as an alternative notation for dw_s. It is important to realise that the integral on the right hand side **cannot be interpreted as a Lebesgue-Stieltjes integral**.

2.1 Example: Wiener process as integrand

Namely take

$$f(w_s) = w_s$$

and suppose to define the integral as

$$\int_0^t w_s^{(\theta)}\, dw_s = \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_k \in \mathsf{p}} w_{\theta_k} \left(w_{t_k} - w_{t_{k-1}}\right) \tag{2.3}$$

As n increases the $\{t_k\}_{k=1}^{n}$ describe sequences of refining partitions of the interval $[0, t]$. The point θ_k is chosen arbitrarily in $[t_{k-1}\,, t_k]$:

$$\theta_k = s\, t_k + (1 - s)\, t_{k-1} \qquad \forall\, s \in [0, 1] \tag{2.4}$$

For ordinary Lebesgue-Stieltjes integrals the right hand side of (2.3) is independent of way θ_k is sampled. In the present case, we instead have

- $\mathbb{L}^2(\Omega)$-convergence of the sum

$$\sum_{t_k \in \mathsf{p}} w_{\theta_k} (w_{t_k} - w_{t_{k-1}}) = \sum_{t_k \in \mathsf{p}} w_{\theta_k} (w_{t_k} - w_{\theta_k}) + \sum_{t_k \in \mathsf{p}} w_{\theta_k} (w_{\theta_k} - w_{t_{k-1}})$$
$$= \sum_{t_k \in \mathsf{p}} \left[\frac{w_{t_k}^2 - w_{\theta_k}^2}{2} - \frac{(w_{t_k} - w_{\theta_k})^2}{2} \right] + \sum_{t_k \in \mathsf{p}} \left[\frac{w_{\theta_k}^2 - w_{t_{k-1}}^2}{2} + \frac{(w_{\theta_k} - w_{t_{k-1}})^2}{2} \right] \tag{2.5}$$

whence it follows

$$\sum_{t_k \in \mathsf{p}} w_{\theta_k} (w_{t_k} - w_{t_{k-1}}) = \frac{w_{t_n}^2}{2} - \sum_{k} \left[\frac{(w_{t_k} - w_{\theta_k})^2}{2} - \frac{(w_{\theta_k} - w_{t_{k-1}})^2}{2} \right]$$

and in $\mathbb{L}^2(\Omega)$ sense

$$\int_0^t w_s^{(\theta)} \, dw_s = \frac{w_t^2}{2} - \frac{t\,(1 - 2\,s)}{2}$$

- Average:

$$\mathrm{E} \int_0^t w_s^{(\theta)} \, dw_s = \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_k \in \mathsf{p}} \mathrm{E} \left\{ w_{\theta_k} \, w_{t_k} - w_{\theta_k} \, w_{t_{k-1}} \right\}$$
$$= \sum_{t_k \in \mathsf{p}} (\theta_k - t_{k-1}) = \sum_{t_k \in \mathsf{p}} s(t_k - t_{k-1}) = \sigma^2 \, s$$

3 Ito integral

Let suppose that ξ_t is a stochastic process satisfying the properties

1. mean square integrability:

$$\mathrm{E} \int_0^t ds \, \xi_s^2 \, < \, \infty$$

2. Non anticipating: ξ_t may depend only on w_s with $s \leq t$. As a consequence ξ_t and dw_t are **independent variables**

$$\mathrm{E} \, \xi_t \, dw_t = \mathrm{E} \xi_t \, \mathrm{E} dw_t = 0$$

Definition 3.1. *For any stochastic process ξ_t satisfying the above two properties we can define the Ito integral*

$$\int_0^t dw_s \, \xi_s := \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_k \in \mathsf{p}} \xi_{t_{k-1}} (w_{t_k} - w_{t_{k-1}}) \tag{3.1}$$

Note that the approximating sums

$$I_n = \sum_{t_k \in \mathsf{p}} \xi_{t_{k-1}} (w_{t_k} - w_{t_{k-1}}) \tag{3.2}$$

are defined in the Ito prescription by setting s to zero in (4.1). The convergence of (3.1) has to be understood in the mean square sense i.e.

$$\mathrm{E} \, (I_n - I_m)^2 \stackrel{n,m \uparrow \infty}{\to} 0$$

The definition (3.1) entails

i the *martingale* property

$$\mathrm{E}\int_0^t dw_s\,\xi_s = 0 \tag{3.3}$$

ii the *mean square integrability* property

$$\mathrm{E}(\int_0^t dw_s\,\xi_s)^2 = \int_0^t ds\,\mathrm{E}\,\xi_s^2 = \mathrm{E}\,\int_0^t ds\,\xi_s^2 \tag{3.4}$$

Namely

$$\mathrm{E}(\int_0^t dw_s\,\xi_s)^2 = \mathrm{E}\lim_{|\mathsf{p}|\downarrow 0}\sum_{t_k,t_l\in\mathsf{p}}(w_{t_{k+1}}-w_{t_k})(w_{t_{l+1}}-w_{t_l})\,\xi_{t_k}\,\xi_{t_l} \tag{3.5}$$

As by hypothesis ξ_t is non anticipating

$$\mathrm{E}\left\{(w_{t_{k+1}}-w_{t_k})(w_{t_{l+1}}-w_{t_l})\,\xi_{t_k}\,\xi_{t_l}\right\} = \delta_{k\,,l}(t_{k+1}-t_k)\,\mathrm{E}\,\xi_{t_k}^2 \tag{3.6}$$

Hence upon inverting the limit and expectation value operation

$$\mathrm{E}(\int_0^t dw_s\,\xi_s)^2 = \lim_{|\mathsf{p}|\downarrow 0}\sum_{t_k,t_l\in\mathsf{p}}\delta_{k\,l}(t_{k+1}-t_k)\,\mathrm{E}\,\xi_{t_k}^2 = \int_0^t dt\,\mathrm{E}\,\xi_t^2 \tag{3.7}$$

which yields the claim. Note that working directly in the continuum limit the above manipulations imply the rule

$$\mathrm{E}\,(dw_s dw_{s'}) = ds'\,\delta(s-s') \tag{3.8}$$

Furthermore since

$$\sum_{k=1}^n w_{t_{k-1}}(w_{t_k}-w_{t_{k-1}}) = \sum_{t_k\in\mathsf{p}}\frac{w_{t_k}^2-w_{t_{k-1}}^2}{2} - \sum_{t_k\in\mathsf{p}}\frac{(w_{t_k}-w_{t_{k-1}})^2}{2}$$

in the mean square sense we can conclude

$$\int_0^t dw_s\,w_s = \frac{w_t^2}{2}-\frac{t}{2}$$

at variance with what expected from the ordinary rules of differential calculus. The origin of the discrepancy from ordinary calculus stems from

$$dw_t \sim O(\sqrt{dt})$$

Example 3.1 (*Non-anticipative vs anticipative*)**.** Let w_t a Wiener process for all $t\geq 0$, the function

$$f(t) = \begin{cases} 0 \quad \text{if} & \max\limits_{0\leq s\leq t} w_s \leq 1 \\ 1 \quad \text{if} & \max\limits_{0\leq s\leq t} \;\; > 1 \end{cases}$$

is *non-anticipative* as it depends on the Wiener process up to the time t when the function is evaluated. On the other hand for any $T>t$ the function

$$g(t) = \begin{cases} 0 \quad \text{if} & \max\limits_{0\leq s\leq T} w_s \leq 1 \\ 1 \quad \text{if} & \max\limits_{0\leq s\leq T} w_s \; > 1 \end{cases}$$

is *anticipative* as it depends on realizations of the Wiener process for times s posterior to the sampling time t.

Example 3.2 (*Exponential process*). Let us consider the process

$$\xi_t = e^{\lambda\, w_t - \frac{\lambda^2\, t}{2}}\, \xi_o \tag{3.9}$$

by Ito lemma we have

$$d\xi_t = \lambda\, dw_t\, e^{\lambda\, w_t - \frac{\lambda^2\, t}{2}}\, \xi_o = \lambda\, \xi_t dw_t$$

If we recast the Ito differential into Doob-Meyer form we find

$$\xi_t = \xi_o + \lambda \int_0^t dw_s\, \xi_s$$

The exponential process does not have bounded variation component.

4 The Stratonovich integral

We have seen that for

$$\theta_k = s\, t_k + (1 - s)\, t_{k-1} \qquad \forall\, s\, \in\, [0, 1] \tag{4.1}$$

the sum

$$\sum_{t_k \in \mathsf{p}} w_{\theta_k}\, (w_{t_k} - w_{t_{k-1}}) = \frac{w_{t_n}^2}{2} - \sum_k \left[\frac{(w_{t_k} - w_{\theta_k})^2}{2} - \frac{(w_{\theta_k} - w_{t_{k-1}})^2}{2} \right]$$

in $\mathbb{L}^2(\Omega)$ converges to

$$\int_0^t w_s^{(\theta)}\, dw_s = \frac{w_t^2}{2} - \frac{t\,(1 - 2\, s)}{2}$$

Choosing $s = 1/2$ the second term on the right hand side disappears and we recover the result from ordinary calculus. The example suggests to define the Fisk-Stratonovich integral

$$\int_0^t dw_s \diamond \xi_s := \lim_{|\mathsf{p} \downarrow 0|} \sum_{t_k \in \mathsf{p}} \xi_{\frac{t_{k-1} + t_k}{2}}\, (w_{t_k} - w_{t_{k-1}}) \tag{4.2}$$

Note that

$$\xi_{\frac{t_{k+1} + t_k}{2}} - \frac{\xi_{t_{k+1}} + \xi_{t_k}}{2} =$$
$$\frac{\xi_{t_{k+1} - \frac{t_{k+1} - t_k}{2}} - \xi_{t_{k+1}}}{2} + \frac{\xi_{t_k + \frac{t_{k+1} - t_k}{2}} - \xi_{t_k}}{2} = O(\xi_{t_{k+1}} - \xi_{t_k})^2$$

Thus we can equivalently write

$$\int_0^t dw_s \diamond \xi_s := \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_k \in \mathsf{p}} \frac{\xi_{t_{k+1}} + \xi_{t_k}}{2}\, (w_{t_k} - w_{t_{k-1}})$$

As in the Ito case the limit converges in mean square sense. At variance with the Ito case, the integrand in the definition (4.2) is anticipating:

$$\mathrm{E}\, \xi_t \diamond dw_t \neq \mathrm{E}\xi_t\, \mathrm{E}dw_t = 0$$

Thus the martingale property of the Ito integral is lost. To appreciate the advantage of the definition consider

$$\int_0^t dw_s \diamond w_s = \lim_{|\mathsf{p}| \downarrow 0} \sum_{t_k \in \mathsf{p}} \frac{(w_{t_k} + w_{t_{k-1}})(w_{t_k} - w_{t_{k-1}})}{2} = \frac{w_t^2}{2} \tag{4.3}$$

in agreement with the rules of *ordinary differential calculus*. The example illustrates the general situation.

4.1 Relation with the Ito differential

Let us consider

$$\xi_t = g(\chi_t, t) \tag{4.4}$$

with

$$d\chi_t = b(\chi_t, t)\, dt + \sigma(\chi_t, t)\, dw_t \tag{4.5}$$

then by Ito lemma we can write

$$d\xi_t = dg(\chi_t, t) = dt \left\{ \partial_t + b_t\, \partial_{\chi_t} + \frac{\sigma^2}{2} \partial^2_{\chi_t} \right\} g + dw_t\, \sigma_t\, \partial_{\chi_t} g \tag{4.6}$$

and use the this result to establish the relation between the Fisk-Stratonovich and the Ito integral. Namely given a non-anticipating process η_t we can couch the definition of the Fisk-Stratonovich integral into the form

$$\int_0^t dw_s \diamond \eta_s = \lim_{|\mathsf{p}|\downarrow 0} \sum_{t_k \in \mathsf{p}} \left\{ \eta_{t_{k-1}}(w_{t_k} - w_{t_{k-1}}) + \frac{(\eta_{t_{k-1}} - \eta_{t_k})(w_{t_k} - w_{t_{k-1}})}{2} \right\}$$

In the literature the latter equality is sometimes written in the continuum limit as

$$\int_0^t dw_s \diamond \eta_s = \int_0^t dw_s\, \eta_s + \langle \eta\, , w \rangle_t$$

where $\langle \xi\, , w \rangle_t$ is *quadratic co-variation* of the processes ξ_t and w_t. The essential point is that in the limit (which converges in the mean square sense under our hypotheses) the quadratic co-variation receives finite contributions only from the term proportional to the increment of the Wiener process

$$dw_t \sim O(\sqrt{dt}) \qquad \Rightarrow \qquad dw_t^2 \sim O(dt) \tag{4.7}$$

In such a case if

$$\eta_t = f(\chi_t, t)$$

we find

$$\int_0^t dw_s \diamond f(\chi_s, s) = \int_0^t dw_s\, f(\chi_s, s) + \frac{1}{2} \int_0^t ds\, \sigma(\chi_s, s)\, \partial_{\chi_s} f(\chi_s, s) \tag{4.8}$$

In particular for

$$f(\chi_t, t) = \sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t)$$

we obtain

$$\begin{aligned} &dw_t \diamond [\sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t)] = \\ &\qquad dw_t\, \sigma(\chi_t, t) \partial_{\chi_t} g(\chi_t, t) + \frac{dt}{2}\, \sigma(\chi_t, t)\, \partial_{\chi_t} [\sigma(\chi_t, t)\, \partial_{\chi_t} g(\chi_t, t)] \end{aligned}$$

which allows us to write

$$d\xi_t = dg(\chi_t, t) = dt \left\{ \partial_t + \left[b - \frac{\sigma}{2} (\partial_{\chi_t} \sigma) \right] \partial_{\chi_t} \right\} g + dw_t \diamond [\sigma \partial_{\chi_t} g] \tag{4.9}$$

As expected, the right hand side does not include any-longer a second derivative of g, the hallmark of Ito lemma. The function g is, however, transported by the Stratonovich stochastic differential

$$d\xi_t = dt \left[b_t - \frac{1}{2} (\sigma \partial_{\chi_t} \sigma)_t \right] + dw_t \sigma_t$$

4.2 Examples

- Consider the process

$$\xi_t = w_t^2$$

In such a case the role of the process χ_t of the previous section is played by the Brownian motion itself

$$\chi_t = w_t \qquad \Rightarrow \qquad d\chi_t = dw_t$$

i.e. $b = 0$ and $\sigma = 1$ in (4.5). Ito lemma yields

$$d\xi_t = dg(w_t) = 2\, w_t\, dw_t + dt$$

The differential admits the equivalent Stratonovich representation

$$d\xi_t = 2\, w_t \diamond dw_t$$

with again $\chi_t = w_t$.

- Consider now

$$\xi_t = \chi_t^2$$

with

$$d\chi_t = \chi_t\, dt + \chi_t\, dw_t \tag{4.10}$$

This case corresponds to

$$b = \sigma = \chi_t$$

in (4.5). It follows by Ito lemma

$$d\xi_t = 3\chi_t^2 dt + 2\, \chi_t^2\, dw_t$$

On the other hand (4.10) admits the Stratonovich representation

$$d\chi_t = \frac{\chi_t}{2}\, dt + \chi_t \diamond dw_t$$

whence

$$d\xi_t = \chi_t^2 dt + 2\, \chi_t^2 \diamond dw_t \tag{4.11}$$

References

[1] L. C. Evans. An introduction to stochastic differential equations. UC Berkeley, Lecture Notes, 2006.

[2] C. W. Gardiner. *Handbook of stochastic methods for physics, chemistry and the natural sciences*, volume 13 of *Springer series in synergetics*. Springer, 2 edition, 1994.