1 Introduction

These notes follow chapter 6 of [1].

2 Stopping time

Definition 2.1 (*Stopping time*)**.** *A random variable*

$$\tau\colon \Omega \to [0\,,\infty]$$

is called a stopping time with respect to a filtration of σ-algebras $\{\mathcal{F}_t \,|\, t \geq 0\}$ provided

$$\{\tau \leq t\} \in \mathcal{F}_t \qquad\qquad \textit{for all} \quad t \geq 0$$

In other words, the set of all $\omega \in \Omega\ \tau(\omega) \leq t$ is $\mathcal{F}_t$-measurable. The stopping time τ is allowed to take on the value $+\infty$, and also that any constant $\tau = t_o$ is a stopping time. Furthermore it enjoys the following properties

Proposition 2.1 (*Properties of a stopping time*)**.** *Let τ_1 and τ_2 stopping times with respect to $\{\mathcal{F}_t \,|\, t \geq 0\}$. Then*

i $\{\tau < t\} \in \mathcal{F}_t$ *and* $\{\tau = t\} \in \mathcal{F}_t$ *for all* $t \geq 0$

ii $\tau_1 \wedge \tau_2$ *and* $\tau_1 \vee \tau_2$ *are stopping times*

Proof. We set

$$\{\tau < t\} = \bigcup_{k=1}^{\infty} \left\{\tau \leq t - \frac{1}{k}\right\}$$

i.e. $\{\tau < t\}$ occurs if there exists a $k \geq 1$ such that the event $\{\tau \leq t - 1/k\}$ occurs. But

$$\{\tau \leq t - 1/k\} \in \mathcal{F}_{t-\frac{1}{k}} \subseteq \mathcal{F}_t$$

Similarly

$$\{\tau_1 \wedge \tau_2 \leq t\} = \{\tau_1 \leq t\} \cup \{\tau_2 \leq t\} \in \mathcal{F}_t$$

and

$$\{\tau_1 \vee \tau_2 \leq t\} = \{\tau_1 \leq t\} \cap \{\tau_2 \leq t\} \in \mathcal{F}_t$$

□

The following theorem evinces the relevance of stopping times for the study of stochastic differential equations

Theorem 2.1. *Let $\boldsymbol{\xi}_t$ solution of the stochastic differential equation*

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t, t)\,dt + \mathsf{A}(\boldsymbol{\xi}_t, t) \cdot\, d\boldsymbol{w}_t$$

$$\boldsymbol{\xi}_{t_o} = \boldsymbol{x}_o$$

satisfying the hypotheses of the theorem of existence and uniqueness. Let also $\mathbb{A}$ be a non-empty open or closed subset of $\mathbb{R}^d$. Then

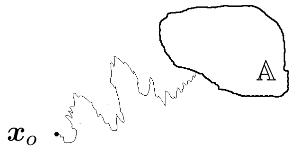
$$\tau := \inf\,\{t \,|\, \xi_t \in \mathbb{A}\}$$

is a stopping time with the convention $\tau = \infty$ if $\boldsymbol{\xi}_t \notin \mathbb{A}$ for all t.

Proof. Let $t \geq 0$ we need to show that $\{\tau \leq t\} \in \mathcal{F}_t$. To that goal we introduce the sequence $\{t_i\}_{i=1}^{\infty}$ dense on $\mathbb{R}_+$ and consider separately the cases when $\mathbb{A}$ is close and open.

- $\mathbb{A}$ is open. The event that there exists a t_i less than t such that $\boldsymbol{\xi}_{t_i}$ belongs to $\mathbb{A}$ reads

$$\{\tau \leq \, t\} = \bigcup_{t_i \leq t} \{\boldsymbol{\xi}_{t_i} \in \mathbb{A}\}$$



  and is therefore the union of events belonging to $\mathcal{F}_t$, thus proving the claim.

- $\mathbb{A}$ is closed. Let

$$d\,(\boldsymbol{x}, \mathbb{A}) := \mathrm{distance}\,(\boldsymbol{x}, \mathbb{A})$$

  and define the open sets

$$\mathbb{A}_n = \left\{ \boldsymbol{x} \,|\, d(\boldsymbol{x}\,, \mathbb{A}) < \frac{1}{n} \right\}$$

  The event

$$\{\tau \leq t\} = \bigcap_{k=1}^{\infty} \bigcup_{t_i \leq t} \{\boldsymbol{\xi}_{t_i} \in \mathbb{A}_n\}$$

  also belongs to $\mathcal{F}_t$ as the $\left\{\boldsymbol{\xi}_{t_i} \in \mathbb{A}_n\right\}$'s do.

□

Remark 2.1. The random variable

$$\tilde{\tau} = \sup\{t \,|\, \xi_t \in \mathbb{A}\}$$

is not in general a stopping time as in general it is not $\mathcal{F}_t$ measurable but may depend on the history of $\boldsymbol{\xi}$ for times later than t.

3 Applications of the stopping time

Let $\boldsymbol{\phi}_t$ be the fundamental solution of the stochastic differential equation

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t\,, t)\; dt + \mathsf{A}\,(\boldsymbol{\xi}_t\,, t) \cdot\; d\boldsymbol{w}_t \tag{3.1}$$

which we assume to globally satisfy the hypotheses of the theorem existence and uniqueness of solutions. In other words for any initial data $(\boldsymbol{x}_o\,, t_o)$ we have that

$$\boldsymbol{\xi}_t = \boldsymbol{\phi}_t\,(\boldsymbol{x}_o\,, t_o) \tag{3.2}$$

for $t \geq t_o$ solves (3.1). To (3.1) also we associate the generator

$$\mathfrak{L}_{\boldsymbol{x}} := \boldsymbol{b}\,(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2}\mathsf{G}\,(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} \tag{3.3}$$

with

$$\mathsf{G} = \mathsf{A}\mathsf{A}^{\dagger} \tag{3.4}$$

3.1 Exit time form a domain

Let $\mathbb{A}$ a smooth bounded open subset of $\mathbb{R}^d$. The stopping time

$$\tau_{\boldsymbol{x},t} = \inf_{t_1} \left\{ t \leq t_1 \leq T \,|\, \boldsymbol{\phi}_{t_1}(\boldsymbol{x}\,, t) \in \partial\mathbb{A} \right\} \tag{3.5}$$

specifies the time when the diffusion process starting from $\boldsymbol{x} \in \mathbb{A}$ at time t exists for the first time the domain $\mathbb{A}$ during a time horizon $[t, T)$.

Proposition 3.1. *Under the above hypotheses, for any $\boldsymbol{x} \in \mathbb{A}$ we have*

$$\mathrm{E}\left(\tau_{\boldsymbol{x},t} \wedge T - t\right) = f\left(\boldsymbol{x}, t\right) \tag{3.6}$$

for

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}}) f\left(\boldsymbol{x}, t\right) = -1 \tag{3.7a}$$

$$f\left(\boldsymbol{x}, \cdot\right)|_{\boldsymbol{x}\in\mathbb{A}} = 0 \tag{3.7b}$$

$$f\left(\cdot, T\right) = 0 \tag{3.7c}$$

More generally for we have

$$\mathrm{E}\left(\tau_{\boldsymbol{x},t} \wedge T - t\right)^n = g_n\left(\boldsymbol{x}, t\right) \tag{3.8}$$

for $g_0(\boldsymbol{x}, t) = 1$

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\, g_n\left(\boldsymbol{x}, t\right) = -n\, g_{n-1}\left(\boldsymbol{x}, t\right) \tag{3.9a}$$

$$g_n\left(\boldsymbol{x}, \cdot\right)|_{\boldsymbol{x}\in\mathbb{A}} = 0 \tag{3.9b}$$

$$g_n\left(\cdot, T\right)|_{\boldsymbol{x}\in\mathbb{A}} = 0 \tag{3.9c}$$

Proof. By Dynkin's formula we have for any sufficiently regular f

$$f\left(\boldsymbol{\phi}_{\tau_{\boldsymbol{x},t}\wedge T}, \tau_{\boldsymbol{x},t} \wedge T\right) = f\left(\boldsymbol{x}, t\right) + \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds\, (\partial_s + \mathfrak{L}_{\boldsymbol{\phi}_s}) f\left(\boldsymbol{\phi}_s, s\right) + \int_t^{\tau_{\boldsymbol{x},t}\wedge T} \left[\mathsf{A}\left(\boldsymbol{\phi}_s, s\right) \cdot d\boldsymbol{w}_s\right] \cdot \partial_{\phi_s} f\left(\boldsymbol{\phi}_s, s\right)$$

If furthermore f satisfies (3.7) then

$$\tau_{\boldsymbol{x},t} \wedge T - t = f\left(\boldsymbol{x}, t\right) + \int_t^{\tau_{\boldsymbol{x},t}\wedge T} \left[\mathsf{A}\left(\boldsymbol{\phi}_s, s\right) \cdot d\boldsymbol{w}_s\right] \cdot \partial_{\phi_s} f\left(\boldsymbol{\phi}_s, s\right)$$

Taking averages proves (3.6). In general, using (3.15b), (3.15c) in Dynkin's formula for $t_1 \geq t$ yields

$$g_n\left(\boldsymbol{\phi}_{t_1}, t_1\right) = -\int_{t_1}^{\tau_{\boldsymbol{x},t}\wedge T} ds\, (\partial_s + \mathfrak{L}_{\boldsymbol{\phi}_s})\, g_n\left(\boldsymbol{\phi}_s, s\right) - \int_{t_1}^{\tau_{\boldsymbol{x},t}\wedge T} \left[\mathsf{A}\left(\boldsymbol{\phi}_s, s\right) \cdot d\boldsymbol{w}_s\right] \cdot \partial_{\phi_s}\, g_n\left(\boldsymbol{\phi}_s, s\right)$$

If we furthermore impose (3.15a) we get into

$$\begin{aligned} g_n(\boldsymbol{x},t) = -n \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds \int_s^{\tau_{\boldsymbol{x},t}\wedge T} ds_1 (\partial_{s_1} + \mathfrak{L}_{\boldsymbol{\phi}_{s_1}}) g_{n-1}\left(\boldsymbol{\phi}_{s_1}, s_1\right) \\ -n \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds \int_s^{\tau_{\boldsymbol{x}}} [\mathsf{A}\left(\boldsymbol{\phi}_{s_1}, s_1\right) \cdot d\boldsymbol{w}_{s_1}] \cdot \partial_{\phi_{s_1}} g_{n-1}\left(\boldsymbol{\phi}_{s_1}, s_1\right) \\ - \int_t^{\tau_{\boldsymbol{x},t}\wedge T} [\mathsf{A}\left(\boldsymbol{\phi}_s, s\right) \cdot d\boldsymbol{w}_s] \cdot \partial_{\phi_s} g_{n-1}\left(\boldsymbol{\phi}_s, s\right) \end{aligned} \tag{3.10}$$

Iterating n-times gives

$$\begin{aligned} g_n(\boldsymbol{x},t) = \Gamma(n+1) \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds_0 \prod_{k=0}^{n-2} \int_{s_k}^{\tau_{\boldsymbol{x},t}\wedge T} ds_{k+1} \int_{s_{n-1}}^{\tau_{\boldsymbol{x},t}\wedge T} ds_l \\ - \sum_{l=1}^{n} \frac{\Gamma(n+1)}{\Gamma(n-l+1)} \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds_0 \prod_{k=0}^{l-2} \int_{s_k}^{\tau_{\boldsymbol{x},t}\wedge T} ds_{k+1} \int_{s_{l-1}}^{\tau_{\boldsymbol{x},t}\wedge T} [\mathsf{A}\left(\boldsymbol{\phi}_{s_l}, s_l\right) \cdot d\boldsymbol{w}_s] \cdot \partial_{\phi_{s_l}} g_{n-1}\left(\boldsymbol{\phi}_{s_l}, s_l\right) \\ - \int_t^{\tau_{\boldsymbol{x},t}\wedge T} [\mathsf{A}\left(\boldsymbol{\phi}_s, s\right) \cdot d\boldsymbol{w}_s] \cdot \partial_{\phi_s} g_{n-1}\left(\boldsymbol{\phi}_s, s\right) \end{aligned} \tag{3.11}$$

Taking the average finally yields

$$g_n(\boldsymbol{x},t) = \Gamma(n+1)\mathrm{E} \int_t^{\tau_{\boldsymbol{x},t}\wedge T} ds_0 \prod_{k=0}^{n-2} \int_{s_k}^{\tau_{\boldsymbol{x},t}\wedge T} ds_{k+1} \int_{s_{n-1}}^{\tau_{\boldsymbol{x},t}\wedge T} ds_l = \mathrm{E}(\tau_{\boldsymbol{x},t} \wedge T - t)^n$$

whence the claim. □

Some observations are in order.

- The boundary conditions associated to (3.15) admit a direct interpretation.
  - (3.15b) states that if the process starts from the boundary the time it takes to reach them is (tautologically) zero.
  - (3.15c) states that if the process starts at time $t = T$ then the random variable

    $$\tau_{\boldsymbol{x},T} \wedge T - T = 0 \tag{3.12}$$

    by construction.
- If the drift and diffusion vector fields are time-independent, time translation invariance is broken only by the final condition. Hence we must have (3.15)

  $$\mathrm{E}(\tau_{\boldsymbol{x},t} \wedge T - t)^n = g_n(\boldsymbol{x}, t; T) = g_n(\boldsymbol{x}, 0; T - t) \tag{3.13}$$

- For an infinite time horizon

  $$\lim_{T\uparrow\infty} (\tau_{\boldsymbol{x},t} \wedge T - t) = \bar{\tau}_{\boldsymbol{x}} \tag{3.14}$$

  Namely by (3.13) the solution of (3.15) must converge to a time independent one solving on its turn the problem

  $$\left[\boldsymbol{b}(\boldsymbol{x}) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2}\mathsf{G}(\boldsymbol{x}) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}\right] g_n(\boldsymbol{x}) = -n\, g_{n-1}(\boldsymbol{x}) \tag{3.15a}$$

$$g_n\left(\boldsymbol{x},\cdot\right)|_{\boldsymbol{x}\in\mathbb{A}} = 0 \tag{3.15b}$$

$$g_0\left(\boldsymbol{x},t\right) = 1 \tag{3.15c}$$

It is possible to recover the above results starting from the forward Kolmogorov (Fokker-Planck) equation. Consider for any $\boldsymbol{x}_o \in \mathbb{A}$ the problem with *absorbing boundary conditions*

$$\partial_t p + \partial_{\boldsymbol{x}} \cdot (\,\boldsymbol{b}\,p) = \frac{1}{2}\partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} : (\mathsf{G}p) \tag{3.16a}$$

$$p|_{\boldsymbol{x}\in\partial\mathbb{A}} = 0 \tag{3.16b}$$

$$\lim_{t\downarrow t_o} p = \delta^{(d)}(\boldsymbol{x}-\boldsymbol{x}_o) \tag{3.16c}$$

The interpretation of absorbing boundary conditions is of removing from the transition probability all those trajectories that for times $s \in [t_o, t]$ reached the boundary. Therefore

$$P\left(\tau_{\boldsymbol{x}_o,t_o} \geq t\right) = \int_{\mathbb{A}} d^d x\, p_{\boldsymbol{\xi}}\left(\boldsymbol{x},t\,|\,\boldsymbol{x}_o,t_o\right)$$

whence we infer

$$p_{\tau_{\boldsymbol{x}_o,t_o}}(t) = -\partial_t \int_{\mathbb{A}} d^d x\, p_{\boldsymbol{\xi}}\left(\boldsymbol{x},t\,|\,\boldsymbol{x}_o,t_o\right)$$

It follows immediately that

$$\mathrm{E}(\tau_{\boldsymbol{x}_o,t_o} \wedge T - t_o)^n = \int_{t_o}^{T} dt\,(t-t_o)^n\, p_{\tau_{\boldsymbol{x}_o,t_o}}(t) + (T-t)^n \int_{T}^{\infty} dt\, p_{\tau_{\boldsymbol{x}_o,t_o}}(t)$$

whence it is straightforward to recover the equation for the moments of the stopping time. Namely if we differentiate with respect to t_o

$$\partial_{t_o}\mathrm{E}(\tau_{\boldsymbol{x}_o,t_o} \wedge T - t_o)^n = -n \int_{t_o}^{T} dt\,(t-t_o)^{n-1}\, p_{\tau_{\boldsymbol{x}_o,t_o}}(t)$$
$$-n\,(T-t)^{n-1} \int_{T}^{\infty} dt\, p_{\tau_{\boldsymbol{x}_o,t_o}}(t) + \mathfrak{L}_{\boldsymbol{x}_o}\mathrm{E}(\tau_{\boldsymbol{x}_o,t_o} \wedge T - t_o)^n \tag{3.17}$$

Inspection of the result allows us to recognize that

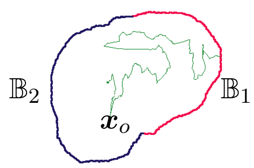
$$\partial_{t_o} g_n(\boldsymbol{x}_o,t_o) = -n\, g_{n-1}(\boldsymbol{x}_o,t_o) - \mathfrak{L}_{\boldsymbol{x}_o} g_n(\boldsymbol{x}_o,t_o) \tag{3.18}$$

which is the result we set out to obtain.

3.2 Hitting one part of a boundary first

Suppose now that the boundary $\partial\mathbb{A}$ of a $\mathbb{A}$ a smooth, bounded, and open subset of $\mathbb{R}^d$ can be decomposed as

$$\partial\mathbb{A} = \mathbb{B}_1 + \mathbb{B}_2$$



with $\mathbb{B}_i$ $i = 1, 2$ smooth. To any $\boldsymbol{x}_o \in \mathbb{A}$ we can associate the stopping time

$$\tau_{\mathbb{B}_i|\boldsymbol{x},t} = \inf_{t_1} \left\{ t \leq t_1 \leq T \,|\, \boldsymbol{\phi}_{t_1}(\boldsymbol{x}, t) \in \mathbb{B}_i \right\} \qquad i = 1, 2 \tag{3.19}$$

through the mapping defined by the fundamental solution of (3.1).

Proposition 3.2. *The probability that $\boldsymbol{\phi}_{t_1}(\boldsymbol{x}, t_1)$ hits first $\mathbb{B}_1$ is specified by the solution of*

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\, u(\boldsymbol{x}, t) = 0 \tag{3.20a}$$

$$u(\boldsymbol{x}, \cdot)\,|_{\boldsymbol{x} \in \mathbb{B}_1} = 1 \qquad \& \qquad u(\boldsymbol{x}, \cdot)\,|_{\boldsymbol{x} \in \mathbb{B}_2} = 0 \tag{3.20b}$$

$$u(\boldsymbol{x}, T)\,|_{\boldsymbol{x} \in \mathbb{B}_1} = \begin{cases} 1 & \forall\, \boldsymbol{x} \in \mathbb{B}_1 \\ 0 & \forall\, \boldsymbol{x} \in \mathbb{A} \cup \mathbb{B}_2 \end{cases} \tag{3.20c}$$

Proof. By Dynkin's formula we have

$$u\left(\boldsymbol{\phi}_{t_1}(\boldsymbol{x}, t), t_1\right) = u(\boldsymbol{x}, t) + \int_t^{t_1} ds\, (\partial_s + \mathfrak{L}_{\boldsymbol{\phi}_s})\, u(\boldsymbol{\phi}_s, s) + \int_0^t [\mathsf{A}(\boldsymbol{\phi}_s, s) \cdot d\boldsymbol{w}_s] \cdot \partial_{\phi_s^i} u(\boldsymbol{\phi}_s, s)$$

Upon setting $t_1 = \tau_{\mathbb{B}_1|\boldsymbol{x},t}$ and requiring $u(\boldsymbol{x}, t)$ to satisfy (3.20), the average of the Dynkin's formula above yields

$$P\left(\tau_{\mathbb{B}_1|\boldsymbol{x},t} \leq T \wedge \tau_{\mathbb{B}_2|\boldsymbol{x},t}\right) = u(\boldsymbol{x}, t)$$

□

Again the boundary conditions admit a direct interpretation

- (3.20b) means that if the process starts for $t < T$ from $\mathbb{B}_1$ or $\mathbb{B}_2$ the event is certain.
- (3.20c) means that if the process starts for $t = T$ the event is also certain because $\mathbb{B}_1$ can be reached for times less than T only if the diffusion starts from $\mathbb{B}_1$.

4 Recurrence of the Wiener process

Let $\boldsymbol{w}_t$ a d-dimensional Wiener motion

$$\xi_t = ||\boldsymbol{w}_t^2||$$

then

$$d\xi_t = d\, dt + 2\sqrt{\xi_t}\frac{\boldsymbol{w}_t \cdot d\boldsymbol{w}_t}{||\boldsymbol{w}_t||}$$

The stochastic process

$$\eta_t = \int_0^t \frac{\boldsymbol{w}_s \cdot d\boldsymbol{w}_s}{||\boldsymbol{w}_s||}$$

enjoys the following properties

- Vanishing first moment

$$\mathrm{E}\,\eta_t \;= 0$$

- Correlation function coinciding with that of the Wiener process

$$\mathrm{E}\,\eta_{t_2}\eta_{t_1} = \int_0^{t_2\wedge t_1} dt \frac{\boldsymbol{w}_t \cdot \boldsymbol{w}_t}{\parallel \boldsymbol{w}_t \parallel^2} = t_2 \wedge t_1$$

- Gaussian statistics: suppose $t_1 \leq t_2 \leq \cdots \leq t_{2n}$

$$\mathrm{E} \prod_{i=1}^{2n} \frac{\boldsymbol{w}_{t_i} \cdot d\boldsymbol{w}_{t_i}}{\parallel \boldsymbol{w}_{t_i} \parallel} = \mathrm{E} \prod_{i=1}^{2n-2} \frac{\boldsymbol{w}_{t_i} \cdot d\boldsymbol{w}_{t_i}}{\parallel \boldsymbol{w}_{t_i} \parallel} \delta(t_{2\,n} - t_{2\,n-1}) dt_{2\,n} = \cdots = \prod_{i=1}^{n} \delta(t_{2\,i} - t_{2\,i-1}) dt_{2\,i} \tag{4.1}$$

  On the other hand

$$\mathrm{E} \prod_{i=1}^{2n+1} \frac{\boldsymbol{w}_{t_i} \cdot d\boldsymbol{w}_{t_i}}{\parallel \boldsymbol{w}_{t_i} \parallel} = 0 \tag{4.2}$$

- Independent increments

$$\eta_{t+t_o} - \eta_{t_o} = \int_{t_o}^{t+t_o} \frac{\boldsymbol{w}_s \cdot d\boldsymbol{w}_s}{||\boldsymbol{w}_s||}$$

Hence η_t is statistically equivalent to a Wiener process:

$$d\xi_t = d\,dt + 2\sqrt{\xi_t} dw_t$$

We can ask whether the Wiener process leaves a ball of radius R around the origin before hitting the origin itself. To answer such question we need to solve for some $0 < \varepsilon < 1$

$$0 = d\,\partial_x u + 2\,x \partial_x^2 u \tag{4.3a}$$

$$u\,(\varepsilon) = 0 \qquad \& \qquad u\,(R) = 1 \tag{4.3b}$$

A straightforward calculation yields

$$u(x) = P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = \begin{cases} \dfrac{R^{1-\frac{d}{2}} - x^{1-\frac{d}{2}}}{R^{1-\frac{d}{2}} - \varepsilon^{1-\frac{d}{2}}} & d \neq 2 \\ \dfrac{\ln R - \ln x}{\ln R - \ln \varepsilon} & d = 2 \end{cases}$$

We observe

$$\lim_{\varepsilon \downarrow 0} P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = \begin{cases} 1 - \left(\dfrac{x}{R}\right)^{1/2} & d = 1 \\ 0 & d \geq 2 \end{cases}$$

In *two dimensions*, nevertheless

$$\lim_{R \uparrow \infty} P\left(\tau_{\varepsilon|x} \leq \tau_{R|x}\right) = 1$$

meaning that the process is *recurrent* in the sense that if $\mathbb{G}$ is any open set

$$P(||\boldsymbol{w}_t||^2 \in \mathbb{G}) = 1$$

References

[1] L. C. Evans. An introduction to stochastic differential equations. UC Berkeley, Lecture Notes, 2006.