Hamilton-Bellman-Jacobi equation

1 Non-homogeneous backward Kolmogorov equation

Let us consider a time continuous Markov process $\{\boldsymbol{\xi}_t\,, t \in [t_{\mathrm{o}}, t_{\mathrm{f}}]\}$

$$\boldsymbol{\xi}_t \colon \Omega \times [t_{\mathrm{o}}, t_{\mathrm{f}}] \mapsto D \tag{1}$$

with generator $\mathfrak{L}$

$$\mathfrak{L}_x = \boldsymbol{b}(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2}\mathsf{A}(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} \tag{2}$$

Let now ψ

$$\psi \colon \mathbb{R}^d \mapsto \mathbb{R} \tag{3}$$

and

$$L \colon \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{R} \tag{4}$$

some smooth functions. Let us then consider the functional

$$V(\boldsymbol{x}, t; T) = \mathrm{E}_{\boldsymbol{x},t}\left\{\int_t^T dt_1\, L(\boldsymbol{\xi}_1, t_1) + \psi(\boldsymbol{\xi}_T)\right\} \tag{5}$$

where as usual

$$\mathrm{E}_{\boldsymbol{x},t}\{\cdot\} := \mathrm{E}\{\cdot \,|\, \boldsymbol{\xi}_t = \boldsymbol{x}\} \tag{6}$$

It is instructive to write (5) explicitly as a functional of the transition probability of the process:

$$V(\boldsymbol{x}, t; T) = \int_t^T dt_2 \int_{\mathbb{R}^d} dx_2\, L(\boldsymbol{x}_2, t_2)\, \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}, t) + \int_{\mathbb{R}^d} d^d x_2\, \psi(\boldsymbol{x}_2)\, \mathrm{p}(\boldsymbol{x}_2, T|\boldsymbol{x}, t) \tag{7}$$

Proposition 1.1. *The function V defined by (7) satisfies the backward non-homogeneous Kolmogorov equation*

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\, V(\boldsymbol{x}, t; T) + L(\boldsymbol{x}, t) = 0 \tag{8a}$$

$$V(\boldsymbol{x}, T; T) = \psi(\boldsymbol{x}) \tag{8b}$$

Proof. The proof follows by direct calculation:

$$\begin{aligned}\partial_t V(\boldsymbol{x}, t; T) = &- \int_{\mathbb{R}^d} dx_2\, L(\boldsymbol{x}_2, t_2)\, \mathrm{p}(\boldsymbol{x}_2, t|\boldsymbol{x}, t) \\ &+ \int_t^T dt_2 \int_{\mathbb{R}^d} dx_2\, L(\boldsymbol{x}_2, t_2)\, (\partial_t \mathrm{p})(\boldsymbol{x}_2, t_2|\boldsymbol{x}, t) + \int_{\mathbb{R}^d} d^d x_2\, \psi(\boldsymbol{x}_2)\, (\partial_t \mathrm{p})(\boldsymbol{x}_2, T|\boldsymbol{x}, t)\end{aligned} \tag{9}$$

The transition probability as a function of the conditioning event satisfies

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})\, \mathrm{p}(\cdot|\boldsymbol{x}, t) = 0 \tag{10a}$$

$$\lim_{t \uparrow t_2} \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}, t) = \delta^{(d)}(\boldsymbol{x}_2 - \boldsymbol{x}) \tag{10b}$$

Hence we obtain

$$\begin{aligned} \partial_t V(\boldsymbol{x}, t; T) &= -L(\boldsymbol{x}, t) \\ &-\mathfrak{L}_x \left\{ \int_t^T dt_2 \int_{\mathbb{R}^d} dx_2\, L(\boldsymbol{x}_2, t_2)\, \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}, t) + \int_{\mathbb{R}^d} d^d x_2\, \psi(\boldsymbol{x}_2)\, \mathrm{p}(\boldsymbol{x}_2, T|\boldsymbol{x}, t) \right\} \end{aligned} \tag{11}$$

which yields the claim. □

For any $t \leq t_1 \leq T$ we can re-write (7) expression as

$$V(\boldsymbol{x}, t; T) = \int_t^{t_1} dt_2 \int_{\mathbb{R}^d} dx_2\, L(\boldsymbol{x}_2, t_2)\, \mathrm{p}(\boldsymbol{x}_2, t_2|\boldsymbol{x}, t) + \int_{\mathbb{R}^d} d^d x_1\, V(\boldsymbol{x}_1, t_1; T)\, \mathrm{p}(\boldsymbol{x}_1, t_1|\boldsymbol{x}, t) \tag{12}$$

which has the same form as (7) on a shorter time horizon $t_1 - t$ and with the replacement

$$\psi(\cdot) \mapsto V(\cdot, t_1; T) \tag{13}$$

As the left hand side in (12) does not depend upon t_1 we must have that

$$\begin{aligned} 0 = \partial_{t_1} V(\boldsymbol{x}, t; T) &= \int_{\mathbb{R}^d} dx_1\, L(\boldsymbol{x}_1, t_1)\, \mathrm{p}(\boldsymbol{x}_1, t_1|\boldsymbol{x}, t) \\ &+ \int_{\mathbb{R}^d} d^d x_1\, \left[(\partial_{t_1} V)(\boldsymbol{x}_1, t_1; T)\, \mathrm{p}(\boldsymbol{x}_1, t_1|\boldsymbol{x}, t) + V(\boldsymbol{x}_1, t_1; T)\, (\partial_{t_1} \mathrm{p})(\boldsymbol{x}_1, t_1|\boldsymbol{x}, t) \right] \end{aligned} \tag{14}$$

The transition probability satisfies as a function of the conditioned even the forward Kolmogorov (Fokker-Planck) equation

$$[(\partial_t - \mathfrak{L}^{\dagger}_{\boldsymbol{x}})\mathrm{p}](\boldsymbol{x}, t|\cdot) = 0 \tag{15}$$

As a consequence a spatial integration by parts in the second integral gives

$$0 = \partial_{t_1} V(\boldsymbol{x}, t; T) = \int_{\mathbb{R}^d} d^d x_1\, \mathrm{p}(\boldsymbol{x}_1, t_1|\boldsymbol{x}, t)\, [L(\boldsymbol{x}_1, t_1) + \partial_{t_1} + \mathfrak{L}_{\boldsymbol{x}_1}]\, V(\boldsymbol{x}_1, t_1) \tag{16}$$

which is self-consistently verified owing to (8). A further consequence is

Proposition 1.2. *Let* $V : \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{R}$ *solution of (8) such that*

$$\mathrm{E} \int_{t_\mathrm{o}}^{t} dt_1\, \mathsf{A}(\boldsymbol{\xi}_t) : (\partial_{\boldsymbol{\xi}_t} V) \otimes (\partial_{\boldsymbol{\xi}_t} V) < \infty \tag{17}$$

then the stochastic process

$$\mu_t = V(\boldsymbol{\xi}_t, t) + \int_{t_\mathrm{o}}^{t} dt_1\, L(\boldsymbol{\xi}_{t_1}, t_1) \tag{18}$$

is a martingale for all $[t_\mathrm{o}, t]$.

By Ito lemma we have

Proof.

$$d\mu_t = dV(\boldsymbol{\xi}_t, t) + L(\boldsymbol{\xi}_t, t) = dt\ \left(\partial_t + \mathfrak{L}_{\boldsymbol{\xi}_t}\right)\ V(\boldsymbol{\xi}_t, t) + [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_t, t) \cdot d\boldsymbol{w}_t] \cdot \partial_{\boldsymbol{\xi}_t} V(\boldsymbol{\xi}_t, t) \tag{19}$$

The function V satisfies by hypothesis (8) hence

$$d\mu_t = [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_t, t) \cdot d\boldsymbol{w}_t] \cdot \partial_{\boldsymbol{\xi}_t} V(\boldsymbol{\xi}_t, t) \tag{20}$$

which shows that V is a local martingale. The integrability condition (17) then guarantees that the integral form of (20) is a stochastic integral well-defined in square mean sense

$$\mu_t - m_o = \int_{t_\mathrm{o}}^{t} [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_{t_1}, t_1) \cdot d\boldsymbol{w}_{t_1}] \cdot \partial_{\boldsymbol{\xi}_{t_1}} V(\boldsymbol{\xi}_{t_1}, t_1) \tag{21}$$

for m_o an integration constant. Hence in the same mean square sense the expected value of μ_t is conserved

$$\mathrm{E}\, \mu_t = m_o \tag{22}$$

and similarly for any $t_\mathrm{o} \leq t_2 \leq t$

$$\mathrm{E}_{\mu_{t_2}}\, \mu_t = \mu_{t_\mathrm{o}} + \int_{t_\mathrm{o}}^{t_2} [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_{t_1}, t_1) \cdot d\boldsymbol{w}_{t_1}] \cdot \partial_{\boldsymbol{\xi}_{t_1}} V(\boldsymbol{\xi}_{t_1}, t_1) = \mu_{t_2} \tag{23}$$

which is the defining property of a martingale. □

The relation between martingales and stochastic integrals is discussed in details in sections **4.3** and **4.6** of [2]. In appendix 3 we recall the definition and the martingale representation theorem.

2 Hamilton-Bellman-Jacobi equation: an heuristic derivation

Let us now consider a class of diffusion processes over the time horizon $[t_\mathrm{o}, t_\mathrm{f}]$ taking values over a state space $\mathbb{S}$ and with with generator of the form

$$\mathfrak{L}_x = \boldsymbol{b}(\boldsymbol{x}, t; \boldsymbol{u}) \cdot \partial_{\boldsymbol{x}} + \frac{1}{2} \mathsf{A}(\boldsymbol{x}, t; \boldsymbol{u}) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}} \tag{24}$$

The notation implies that the drift and the diffusion fields depend upon a vector field $\boldsymbol{u}$. We will refer to $\boldsymbol{u}$ in what follows as the "stochastic control" of the problem. We set out to determine the functional dependence of $\boldsymbol{u}$ upon $\mathbb{S} \times [t_\mathrm{o}, t_\mathrm{f}]$ with respect to the control a functional of the process of the form

$$V(\boldsymbol{x}, t_o; t_f) = \min_{u} \mathrm{E}_{\boldsymbol{x}, t_o} \left\{ U\left(\boldsymbol{\xi}_{t_\mathrm{f}}\right) + \int_{t_o}^{t_f} dt\, L\left(\boldsymbol{\xi}_t, t; \boldsymbol{u}\right) \right\} \tag{25}$$

To fix the terminology with will convene to call

- L the running cost function;
- U the terminal cost function;
- V the value function.

Proceeding in an heuristic fashion, we observe that for any u for which there exists a (non-optimal) diffusion process in the horizon $[t_{\mathrm{o}}, t_{\mathrm{f}}]$ we can re-phrase (25) as

$$V\left(\boldsymbol{x}, t_o; t_f\right) = \min_{u} J(\boldsymbol{x}, t_{\mathrm{o}}; t_{\mathrm{f}}, \boldsymbol{u}) \tag{26a}$$

$$J(\boldsymbol{x}, t; t_{\mathrm{f}}, \boldsymbol{u}) = \int_{t}^{t_f} dt_1 \int_{\mathbb{S}} d^d x_1 L\left(\boldsymbol{x}_1, t_1\right) \mathrm{p}\left(\boldsymbol{x}_1, t_1 | \boldsymbol{x}, t\right) + \int_{\mathbb{S}} d^d x_1\, U\left(\boldsymbol{x}_1\right) \mathrm{p}\left(\boldsymbol{x}_1, t_{\mathrm{f}} | \boldsymbol{x}, t\right) \tag{26b}$$

Note that we suppose that U is independent of $\boldsymbol{u}$. From the analysis of the previous section we expect that the function J satisfies the backward Kolmogorov equation (omitting parametric dependencies)

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}}) J(\boldsymbol{x}, t) + L(\boldsymbol{x}, t) = 0 \tag{27a}$$

$$J(\boldsymbol{x}, t_f) = U(\boldsymbol{x}) \tag{27b}$$

Suppose now that the set of admissible controls $\boldsymbol{u}$ is smoothly parametrized by a scalar quantity ε. If indeed (25) admits a minimum, there must be a value $\varepsilon_\star$ of ε,

$$\boldsymbol{u}'_\star := \left.\frac{d\boldsymbol{u}}{d\varepsilon}\right|_{\varepsilon=\varepsilon_\star} \tag{28}$$

such that

$$J'_\star(\boldsymbol{x}, t) := (\boldsymbol{u}' \cdot \partial_{\boldsymbol{u}} J)(\boldsymbol{x}, t)|_{\varepsilon=\varepsilon_\star} = 0 \qquad \forall\, (\boldsymbol{x}, t) \in \mathbb{S} \times [t_{\mathrm{o}}, t_{\mathrm{f}}] \tag{29}$$

independently of $\boldsymbol{u}'$. In order to identify the critical point we can take the variation of (27) which yields

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}}) J'(\boldsymbol{x}, t) + [\boldsymbol{b}'(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \mathsf{A}'(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}] J(\boldsymbol{x}, t) + L'(\boldsymbol{x}, t) = 0 \tag{30a}$$

$$J'(\boldsymbol{x}, t_{\mathrm{f}}) = 0 \tag{30b}$$

As the equation for J' is linear, for arbitrary $\boldsymbol{u}$ we have

$$J'(\boldsymbol{x}, t) = \int_{t}^{t_{\mathrm{f}}} dt_1 \int_{\mathbb{S}} d^d x_1 \left\{ [\boldsymbol{b}'(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \mathsf{A}'(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}] J(\boldsymbol{x}, t) + L'(\boldsymbol{x}, t) \right\} \mathrm{p}_\star(\boldsymbol{x}_1, t_1 | \boldsymbol{x}, t) \tag{31}$$

with $\mathrm{p}_\star$ the transition probability of the optimal process. If the drift and diffusion fields are sufficiently regular, the system (30) admits an identically vanishing solution for a non-vanishing $J_\star$ if the non-homogeneous term in (30a) vanishes i.e. if

$$[(\partial_{\boldsymbol{u}} \boldsymbol{b})(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + (\partial_{\boldsymbol{u}} \mathsf{A})(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}] J(\boldsymbol{x}, t) + \partial_{\boldsymbol{u}} L(\boldsymbol{x}, t) = 0 \tag{32}$$

The equation (32) specifies in general the critical values of $\boldsymbol{u}$. In order to determine the minimizer, we should turn to the study of the second variation of J. Around a critical point, the second variation must satisfy

$$(\partial_t + \mathfrak{L}_{\boldsymbol{x}})_\star J''_\star(\boldsymbol{x}, t) + [\boldsymbol{b}''_\star(\boldsymbol{x}, t) \cdot \partial_{\boldsymbol{x}} + \mathsf{A}''_\star(\boldsymbol{x}, t) : \partial_{\boldsymbol{x}} \otimes \partial_{\boldsymbol{x}}] J(\boldsymbol{x}, t) + L''_\star(\boldsymbol{x}, t) = 0 \tag{33a}$$

$$J''_\star(\boldsymbol{x}, t_{\mathrm{f}}) = 0 \tag{33b}$$

which we can re-write as

$$J''_\star(\boldsymbol{x},t) = \int_t^{t_\mathrm{f}} dt_1 \int_{\mathbb{S}} d^d x_1 \left\{[\boldsymbol{b}''_\star(\boldsymbol{x},t)\cdot\partial_{\boldsymbol{x}} + \mathsf{A}''_\star(\boldsymbol{x},t) : \partial_{\boldsymbol{x}}\otimes\partial_{\boldsymbol{x}}]J(\boldsymbol{x},t) + L''_\star(\boldsymbol{x},t)\right\} \mathrm{p}_\star(\boldsymbol{x}_1,t_1|\boldsymbol{x},t) \tag{34}$$

The very interpretation of $\mathrm{p}_\star$ as transition probability imposes that this quantity must positive definite. It follows that the second variation of J is positive definite for an arbitrary variation around the critical point if

$$\boldsymbol{v}\cdot\{[(\partial_{\boldsymbol{u}}\otimes\partial_{\boldsymbol{u}}\boldsymbol{b})(\boldsymbol{x},t)\cdot\partial_{\boldsymbol{x}} + (\partial_{\boldsymbol{u}}\otimes\partial_{\boldsymbol{u}}\mathsf{A})(\boldsymbol{x},t) : \partial_{\boldsymbol{x}}\otimes\partial_{\boldsymbol{x}}]J(\boldsymbol{x},t) + (\partial_{\boldsymbol{u}}\otimes\partial_{\boldsymbol{u}}L)(\boldsymbol{x},t)\}\cdot\boldsymbol{v} \geq 0 \tag{35}$$

for any $\boldsymbol{v}\in\mathbb{S}$. In particular, if drift and diffusion fields are linear in the stochastic control $\boldsymbol{u}$ the condition reduces to the requirement that the running cost be a *convex* function of the control itself

$$\boldsymbol{v}\cdot(\partial_{\boldsymbol{u}}\otimes\partial_{\boldsymbol{u}})L(\boldsymbol{x},t)\cdot\boldsymbol{v}\geq 0 \tag{36}$$

for any $\boldsymbol{v}\in\mathbb{S}$. The conclusion of this heuristic discussion is that the value function (25) must solve the Hamilton-Bellman-Jacobi equation

$$\partial_t V(\boldsymbol{x},t) + \min_{\boldsymbol{u}}\{\mathfrak{L}_{\boldsymbol{x}}V(\boldsymbol{x},t) + L(\boldsymbol{x},t)\} = 0 \tag{37a}$$

$$V(\boldsymbol{x},t_\mathrm{f}) = U(\boldsymbol{x}) \tag{37b}$$

Two observations are in order.

- The key point of the above derivation is that we can determine the optimal control by minimizing ***locally*** at each time step the running cost. The Hamilton-Bellman-Jacobi (37a) equation must therefore admit the interpretation of being the backward Kolmogorov equation of the optimal process.
- The minimum condition in (37a) may admit more than one solution. In such a case, it is necessary to verify a-posteriori which solution indeed corresponds to the optimum.
- In general, even after finding a unique solution of (37) it is still necessary to verify that the critical value of $\boldsymbol{u}$ associated to it indeed corresponds to a well-defined diffusion process with generator

$$(\mathfrak{L}_{\boldsymbol{x}})_\star := \boldsymbol{b}(\boldsymbol{x},t;\boldsymbol{u}_\star)\cdot\partial_{\boldsymbol{x}} + \frac{1}{2}\mathsf{A}(\boldsymbol{x},t;\boldsymbol{u}_\star) : \partial_{\boldsymbol{x}}\otimes\partial_{\boldsymbol{x}} \tag{38}$$

  The conditions that the value function V must satisfy to pass such self-consistence check are specified by *verification theorems*. We will later briefly expound the ideas behind these theorems.

The fact that optimal control of a Markov process stems from a set of local operation is encapsulated in **Bellman's principle** which we can state as the following proposition

Proposition 2.1. *An optimal Markov control over an horizon* $[t_\mathrm{o},t_\mathrm{f}]$ *is specified by the requirement that the value function be of stationary variation for any sub-interval* $[t,t_\mathrm{f}]$ $t_\mathrm{o}\leq t\leq t_\mathrm{f}$ *while holding fixed the state at time t.*

The following calculation further evinces the self-consistence of the heuristic considerations brought forth to substantiate Bellman principle . Namely, assuming a smooth dependence of the diffusion over $\boldsymbol{u}$ and using (27a) we have

$$\begin{aligned}\mathcal{A}'(\boldsymbol{x},t_o;t_f) &= \int_{\mathbb{S}} d^d x_f\, U(\boldsymbol{x}_f)\ \mathrm{p}'(\boldsymbol{x}_f,t|\boldsymbol{x},t_o)\\ &+ \int_{t_o}^{t_f} dt \int_{\mathbb{S}} d^d x_f\ \{-[(\partial_t + \mathfrak{L}_{\boldsymbol{x}_f})J](\boldsymbol{x}_f,t)\ \mathrm{p}'(\boldsymbol{x}_f,t|\boldsymbol{x},t_o) + L'(\boldsymbol{x}_f,t)\ \mathrm{p}(\boldsymbol{x}_f,t|\boldsymbol{x},t_o)\}\end{aligned} \tag{39}$$

We can rewrite this equation as

$$\mathcal{A}'(\boldsymbol{x}, t_o; t_f) = \int_{\mathbb{S}} d^d x_f \, U(\boldsymbol{x}_f) \; \mathrm{p}'(\boldsymbol{x}_f, t_\mathrm{f} | \boldsymbol{x}, t_o)$$
$$-\left\{ \int_{t_o}^{t_f} dt \int_{\mathbb{S}} d^d x_f \, [(\partial_t + \mathfrak{L}_{\boldsymbol{x}_f}) J] (\boldsymbol{x}_f, t) \; \mathrm{p}(\boldsymbol{x}_f, t | \boldsymbol{x}, t_o) \right\}'$$
$$+ \int_{t_o}^{t_f} dt \int_{\mathbb{S}} d^d x_f \left\{ [(\partial_t + \mathfrak{L}_{\boldsymbol{x}_f}) J]' (\boldsymbol{x}_f, t) + L'(\boldsymbol{x}_f, t) \right\} \mathrm{p}(\boldsymbol{x}_f, t | \boldsymbol{x}, t_o) \tag{40}$$

The argument of the third integral vanishes by (30a), whilst after an integration by parts in the second integral we obtain

$$\mathcal{A}'(\boldsymbol{x}, t_o; t_f) = \int_{\mathbb{S}} d^d x_f \, [U(\boldsymbol{x}_f) - J(\boldsymbol{x}_f, t_\mathrm{f})] \, \mathrm{p}'(\boldsymbol{x}_f, t_\mathrm{f} | \boldsymbol{x}, t_o)$$
$$+ J'(\boldsymbol{x}, t_\mathrm{o}) - \int_{\mathbb{S}} d^d x_f \, J'(\boldsymbol{x}_f, t_\mathrm{f}) \, \mathrm{p}(\boldsymbol{x}_f, t_\mathrm{f} | \boldsymbol{x}, t_o)$$
$$- \left\{ \int_{t_o}^{t_f} dt \int_{\mathbb{S}} d^d x_f \, [(-\partial_t + \mathfrak{L}^\dagger_{\boldsymbol{x}_f}) \mathrm{p}] (\boldsymbol{x}_f, t | \boldsymbol{x}, t_o) \, J(\boldsymbol{x}, t) \right\}' \tag{41}$$

If $\mathfrak{L}$ is the generator of a Markov process, the adjoint operation $\mathfrak{L}^\dagger$ specifies the evolution of the probability density

$$(-\partial_t + \mathfrak{L}^\dagger_{\boldsymbol{x}}) p_{\boldsymbol{\xi}}] (\boldsymbol{x}_f, t | \boldsymbol{x}, t_o) = 0 \tag{42}$$

It is here worth emphasizing that whilst Ito lemma always implies that $\mathfrak{L}$ is a differential operator $\mathfrak{L}^\dagger$, instead, is not necessarily a differential operator (see e.g. [1] for classical examples). Taking into account the boundary conditions, the variation finally reduces to

$$\mathcal{A}'(\boldsymbol{x}, t_o; t_f) = J'(\boldsymbol{x}, t_\mathrm{o}) \tag{43}$$

as claimed.

3 Verification theorems and Martingales

Let us consider again the optimization problem (25) and suppose that we know the value function up to a time $t_2 \leq t_\mathrm{f}$, then for any *non-optimal* choice of the control in the interval $[t, t_2)$ we have

$$V(\boldsymbol{\xi}_t, t) \leq J(\boldsymbol{x}, t) := \mathrm{E}_{\boldsymbol{x}, t} \left\{ \int_t^{t_2} dt_1 \, L(\boldsymbol{\xi}_{t_1}, t_1; \boldsymbol{u}) + V(\boldsymbol{\xi}_{t_2}, t_2) \right\} \tag{44}$$

This means that the process

$$\tilde{\mu}_t = V(\boldsymbol{\xi}_t, t) + \int_{t_\mathrm{o}}^t dt_1 \, L(\boldsymbol{\xi}_{t_1}, t_1; \boldsymbol{u}) \tag{45}$$

specified by the sum of the V plus the time integral of the running cost evaluated over a *non-optimal* protocol defines a *sub-martingale*. Namely direct differentiation yields

$$d\tilde{\mu}_t = dt \, \left[(\partial_t + \mathfrak{L}^{[\boldsymbol{u}_\star]}_{\boldsymbol{\xi}_t}) \, V(\boldsymbol{\xi}_t, t) + L(\boldsymbol{\xi}_t, t; \boldsymbol{u}) \right] + [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_t, t) \cdot d\boldsymbol{w}_t] \cdot \partial_{\boldsymbol{\xi}_t} V(\boldsymbol{\xi}_t, t) \tag{46}$$

Thus we see that the drift vanishes if we set the control $\boldsymbol{u}$ equal to its optimal value $\boldsymbol{u}_\star$. In such a case, the sub-martingale becomes a local martingale. We infer that the verification criterium for deciding that the solution V of the

Hamilton Jacobi equation (37) specifies indeed the sought value function for the optimal control problem is that the process

$$\mu_t = \int_{t_o}^{t} [\sqrt{\mathsf{A}}(\boldsymbol{\xi}_{t_1}, t_1) \cdot d\boldsymbol{w}_{t_1}] \cdot \partial_{\boldsymbol{\xi}_{t_1}} V(\boldsymbol{\xi}_{t_1}, t_1) \tag{47}$$

is indeed a martingale, see [2] for further details.

Martingale definition

Definition .1. *A stochastic process $\{\boldsymbol{\xi}_t, t \in \mathbb{R}_+\}$ is a **martingale** if for any t it is integrable,*

$$\mathrm{E} \parallel \boldsymbol{\xi}_t \parallel < \infty \tag{48}$$

and for any $t_1 > 0$

$$\mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\mathcal{F}}_t^{(\xi)}\right\} \equiv \mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\xi}_t\right\} = \boldsymbol{\xi}_t \qquad \text{a.s.} \tag{49}$$

*where $\boldsymbol{\mathcal{F}}_t^{(\xi)}$ is the natural filtration induced by $\boldsymbol{\xi}_t$ (i.e. the information about the process up to time t), and the equality holds almost surely. It is a **sub-martingale** if under the same hypotheses*

$$\mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\mathcal{F}}_t^{(\xi)}\right\} \equiv \mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\xi}_t\right\} \geq \boldsymbol{\xi}_t \qquad \text{a.s.} \tag{50}$$

*and a **super-martingale** if*

$$\mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\mathcal{F}}_t^{(\xi)}\right\} \equiv \mathrm{E}\left\{\boldsymbol{\xi}_{t+t_1} | \boldsymbol{\xi}_t\right\} \leq \boldsymbol{\xi}_t \qquad \text{a.s.} \tag{51}$$

Any stochastic differential equation without drift e.g.

$$d\boldsymbol{\xi}_t = \mathsf{A}(\boldsymbol{\xi}_t, t) \cdot d\boldsymbol{w}_t \tag{52}$$

is said to define a local martingale. It defines a martingale with respect to the filtration of the Wiener process in $[0, t]$ if

$$\mathrm{E} \int_0^t dt_1 \, \boldsymbol{v} \cdot (\mathsf{A}\mathsf{A}^\dagger)(\boldsymbol{\xi}_t, t) \cdot \boldsymbol{v} < \infty \tag{53}$$

for any $\boldsymbol{v} \in \mathbb{R}^d$, (53) being the condition ensuring the existence of the stochastic integral in square mean sense. The converse of this result is the martingale representation theorem

Theorem .1. *Let $\boldsymbol{\xi}_t$ be a martingale with respect to the filtration $\boldsymbol{\mathcal{F}}_T^w$ of the Wiener process such that*

$$\mathrm{E} \parallel \boldsymbol{\xi}_t \parallel^2 < \infty \qquad \forall\, t \leq T \tag{54}$$

Then there exists a unique $\boldsymbol{\mathcal{F}}_T^w$-adapted process A_t verifying (53) such that

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_o + \int_0^t \mathsf{A}_{t_1} \cdot d\boldsymbol{w}_{t_1} \tag{55}$$

The uniqueness of A_t is required modulo a measure zero set in $\mathrm{P} \times \mu_{[0,t]}$ where P is the measure over $\boldsymbol{\xi}_t$ and $\mu_{[0,t]}$ the Lebesgue measure over $[0, t]$.

References

[1] W. Feller. Two singular diffusion problems. *The Annals of Mathematics*, 54(1):173–182, 1951.

[2] R. van Handel. Stochastic calculus and stochastic control. Lecture Notes, Caltech, 2007.